Example: $\int x \tan^{-1}(x)\, dx$

This is a slightly harder problem than any on the test, but something like this might appear on the final. How should we approach this?

Student: Integration by parts.

Great! Because $\tan^{-1}(x)$ is begging to be differentiated to be made simpler. So we choose:

$$u = \tan -1(x), \qquad v' = x,$$
$$u' = \frac{1}{1+x^2}, \qquad v = \frac{x^2}{2}.$$

Integration by parts then gives us:

$$\int x \tan^{-1}(x)\, dx = \frac{x^2}{2} \tan^{-1}(x) - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2}\, dx$$

We're not done yet! We still have to integrate:

$$-\frac{1}{2} \int \frac{x^2}{1+x^2}\, dx.$$

What do we do know?

Student: Trig substitution.

Trig substitution will work, but that's not what I had in mind.

Student: Add and subtract one in the numerator.

That's a good idea. This is an example of a rational expression in which the numerator and denominator have the same degree, so you could use long division to simplify the "improper fraction". An equivalent shortcut is:

$$\begin{aligned}
\frac{x^2}{1+x^2} &= \frac{x^2+1-1}{1+x^2} \\
&= \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \\
&= 1 - \frac{1}{1+x^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\frac{1}{2} \int \frac{x^2}{1+x^2}\, dx &= -\frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \\
&= -\frac{1}{2}x + \frac{1}{2} \tan^{-1} x + C.
\end{aligned}$$

The answer to the original question is then:

$$\begin{aligned}
\int x \tan^{-1}(x)\, dx &= \frac{x^2}{2} \tan^{-1}(x) - \int \frac{x^2}{2} \frac{1}{1+x^2}\, dx \\
&= \frac{x^2}{2} \tan^{-1}(x) + \frac{1}{2}x - \frac{1}{2} \tan^{-1} x + c.
\end{aligned}$$

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