Example: Volume of a Sphere

Let's practice the method of disks by finding the volume of a soccer ball. We'll find the volume of revolution formed by rotating a circle with center at $(a, 0)$ and radius a about the x-axis. This will sweep out a ball of radius a. Our goal is to find out the volume of this ball.

Remember that our integrand will be $dV = \pi y^2\,dx$. In order to use this we need a formula for y in terms of x, so we need the equation of the circle of radius a centered at $(a, 0)$:

$$(x - a)^2 + y^2 = a^2.$$

We need to describe y in terms of x:

$$\begin{aligned}
(x-a)^2 + y^2 &= a^2 \\
y^2 &= a^2 - (x-a)^2 \\
&= a^2 - (x^2 - 2ax + a^2) \\
y^2 &= 2ax - x^2
\end{aligned}$$

We can stop here (without taking any square roots) because the value we need to plug into our formula is y^2. We get:

$$V = \int \pi(2ax - x^2)\,dx.$$

Of course we can't evaluate this integral without knowing the limits of integration. Luckily those limits are easy to find in this example. We're integrating with respect to x. The lowest value of x in our circle is $x = 0$ and the highest is $2a$, so x ranges between 0 and $2a$. Our formula for the volume of a ball becomes:

$$\begin{aligned}
V &= \int_0^{2a} \pi(2ax - x^2)\,dx \\
&= \pi\left(ax^2 - \frac{x^3}{3}\right)\Bigg|_0^{2a} \\
&= \pi\left(4a^3 - \frac{8a^3}{3}\right) - 0 \\
&= \left(\frac{12}{3} - \frac{8}{3}\right)\pi a^3 \\
V &= \frac{4}{3}\pi a^3
\end{aligned}$$

One nice thing about this formula is that we've found more than just the volume of the ball. If we change the upper limit of integration we can also find the volume of a piece sliced off the ball.

$$V(x) \quad = \quad \text{Volume of a chopped off portion of the sphere with width } x$$

$$
\begin{aligned}
&= \int_0^x \pi(2at - t^2)\,dt \\
V(x) &= \pi\left(ax^2 - \frac{x^3}{3}\right)
\end{aligned}
$$

If we plug in $x = a$ we should get half the volume of the sphere:

$$
\begin{aligned}
V(a) &= \text{Volume of a half sphere} \\
&= \pi\left(a^3 - \frac{a^2}{3}\right) \\
&= \pi\left(a^3 - \frac{a^2}{3}\right) \\
&= \pi\frac{2}{3}a^3 \\
&= \frac{1}{2}\left(\frac{4}{3}\pi a^3\right)
\end{aligned}
$$

This is a good way to check our work.

The formula $V(x) = \pi\left(ax^2 - \frac{x^3}{3}\right)$ turns out to be useful in predicting the behavior of particles in a fluid. When large spherical particles are being pushed around by small ones, will they tend to cluster together or to stick to the sides of the container? Finding the answer to this question involves adding the volumes of two slices off a sphere to find the volume of a lens shaped region.