Example of Estimation

Here's an example in which we use the estimation property of integrals: if $f(x) \leq g(x)$ and $a < b$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

The example is the same as one we've already seen. We'll start with an inequality and then integrate it to reach a conclusion about the antiderivatives.

We know that $e^x \geq 1$ for $x \geq 0$; this is our starting place. We integrate this expression, then follow our noses to get the result we're expecting:

$$\begin{aligned} e^x &\geq 1 \quad (x \geq 0) \\ \int_0^b e^x\,dx &\geq \int_0^b 1\,dx \quad (b \geq 0) \\ e^x\big|_0^b &\geq b \quad \text{(area of rectangle with base } b \text{ and height 1.)} \\ e^b - 1 &\geq b \\ e^b &\geq 1 + b \quad (b \geq 0) \end{aligned}$$

Notice that we can still compute the integral if $b < 0$, but in that case e^b is not greater than or equal to 1, and so we can't use the estimation property to conclude that $e^b \geq 1 + b$.

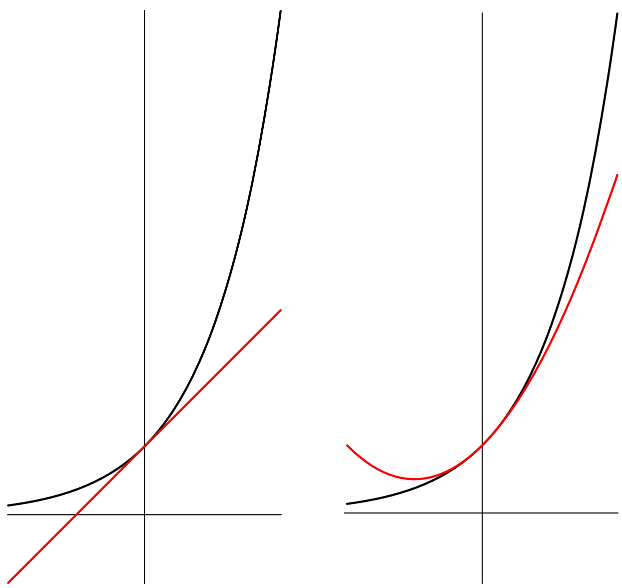

Figure 1: The graphs of e^x (black) compared to $1 + x$ and $1 + x + \frac{x^2}{2}$ (red).

Now we repeat the process starting from the conclusion:

$$\begin{aligned} e^x &\geq 1 + x \quad (x \geq 0) \\ \int_o^b e^x\,dx &\geq \int_0^b (1 + x)\,dx \quad (b \geq 0) \end{aligned}$$

$$\begin{aligned} e^b - 1 &\geq \left(x + \frac{x^2}{2}\right)\Bigg|_0^b \\ e^b - 1 &\geq b + \frac{b^2}{2} \\ e^b &\geq 1 + b + \frac{b^2}{2} \quad (b \geq 0) \end{aligned}$$

In this case, the conclusion is false if $b < 0$.

We can easily keep going with this, producing higher and higher degree interesting polynomial lower bounds for e^x. For example, if we let $b = 1$ in our final conclusion we discover that $e \geq 2\frac{1}{2}$.