Antiderivative of $\tan x \sec^2 x$

Compute $\int \tan x \sec^2 x \, dx$ in two different ways:

a) By substituting $u = \tan x$.

b) By substituting $v = \sec x$.

c) Compare the two results.

Solution

a) Compute $\int \tan x \sec^2 x \, dx$ by substituting $u = \tan x$.

If $u = \tan x$ then $du = \sec^2 x \, dx$ and:

$$\begin{aligned} \int \tan x \sec^2 x \, dx &= \int u \, du \\ &= \frac{1}{2} u^2 + c \\ &= \frac{1}{2} \tan^2 x + c. \end{aligned}$$

b) Compute $\int \tan x \sec^2 x \, dx$ by substituting $v = \sec x$.

If $v = \sec x$ then $dv = \sec x \tan x \, dx$ and:

$$\begin{aligned} \int \tan x \sec^2 x \, dx &= \int \sec x (\tan x \sec x \, dx) \\ &= \int v \, dv \\ &= \frac{1}{2} v^2 + C \\ &= \frac{1}{2} \sec^2 x + C. \end{aligned}$$

c) Compare the two results.

At first glance you may think you made a mistake; it is not true that $\tan^2 x = \sec^2 x$. However, you can see from the graph in Figure 1 that your two answers may only differ by a constant.

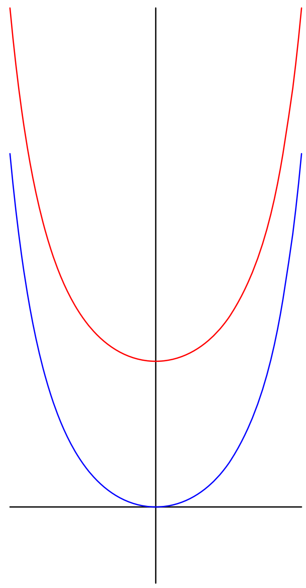

Figure 1: Graphs of $\tan^2 x$ (blue) and $\sec^2 x$ (red).

In fact, that is the case:

$$\begin{aligned} \tan^2 x &= \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{1-\cos^2 x}{\cos^2 x} \\ &= \sec^2 x - 1 \end{aligned}$$

We conclude that $\frac{1}{2}\tan^2 x = \frac{1}{2}\sec^2 x - \frac{1}{2}$ and so the two results are equivalent up to an added constant. Both answers are correct.