Newton's Method

Newton's method is a powerful tool for solving equations of the form $f(x) = 0$.

Example: Solve $x^2 = 5$.

We're going to use Newton's method to find a numerical approximation for $\sqrt{5}$. Any equation that you understand can be solved this way. In order to use Newton's method, we define $f(x) = x^2 - 5$. By finding the value of x for which $f(x) = 0$ we solve the equation $x^2 = 5$.

Our goal is to discover where the graph crosses the x-axis. We start with an initial guess — we'll guess $x_0 = 2$, since $\sqrt{5} \approx \sqrt{4} = 2$. This is not a very good guess; $f(2) = -1$, and we're looking for a number x for which $f(x) = 0$. We'll try to improve our guess.

We pretend that the function is linear, and look for the point where the tangent line to the function at x_0 crosses the x-axis: see Fig. 1. This point $(x_1, 0)$ gives us a new guess at our solution: x_1.

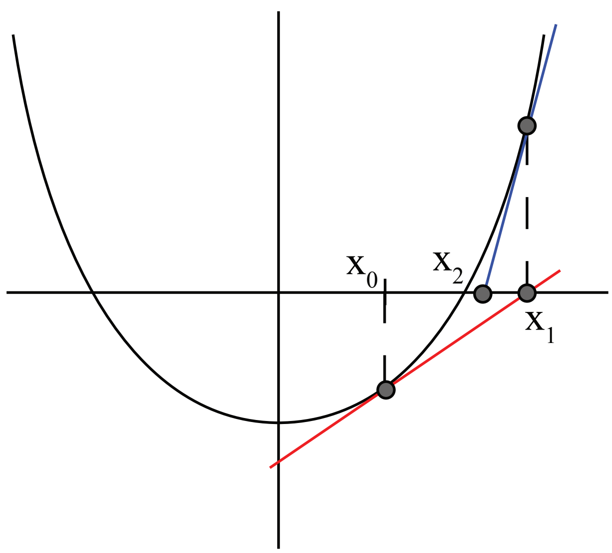

Figure 1: Illustration of Newton's Method

The equation for the tangent line is:

$$y - y_0 = m(x - x_0)$$

When the tangent line intercepts the x-axis $y = 0$, and the x coordinate of that point is our new guess x_1.

$$\begin{aligned} -y_0 &= m(x_1 - x_0) \\ -\frac{y_0}{m} &= x_1 - x_0 \\ x_1 &= x_0 - \frac{y_0}{m} \end{aligned}$$

In terms of f:

$$\begin{aligned} y_0 &= f(x_0) \\ m &= f'(x_0) \end{aligned}$$

because m is the slope of the tangent line to $y = f(x)$ at the point (x_0, y_0). Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The point of Newton's method is that we can improve our new guess by repeating this process. To get our $(n+1)^{st}$ guess we apply this formula to our n^{th} guess:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.}$$

In our example, $x_0 = 2$ and $f(x) = x^2 - 5$. We first calculate $f'(x) = 2x$. Thus,

$$\begin{aligned} x_1 &= x_0 - \frac{(x_0^2 - 5)}{2x_0} = x_0 - \frac{1}{2}x_0 + \frac{5}{2x_0} \\ x_1 &= \frac{1}{2}x_0 + \frac{5}{2x_0} \end{aligned}$$

The main idea is to repeat (iterate) this process:

$$\begin{aligned} x_2 &= \frac{1}{2}x_1 + \frac{5}{2x_1} \\ x_3 &= \frac{1}{2}x_2 + \frac{5}{2x_2} \end{aligned}$$

and so on. The procedure approximates $\sqrt{5}$ extremely well.

Let's see how well this works:

$$\begin{aligned}
x_1 &= \frac{1}{2}2 + \frac{5}{2\cdot 2} \\
&= 1 + \frac{5}{4} \\
&= \frac{9}{4} \\
x_2 &= \frac{1}{2}\frac{9}{4} + \frac{5}{2\frac{9}{4}} \\
&= \frac{9}{8} + \frac{5}{2}\frac{4}{9} \\
&= \frac{9}{8} + \frac{10}{9} \\
&= \frac{161}{72} \\
x_3 &= \frac{1}{2}\frac{161}{72} + \frac{5}{2}\frac{72}{161}
\end{aligned}$$

n	x_n	$\sqrt{5} - x_n$
0	2	2×10^{-1}
1	$\frac{9}{4}$	10^{-2}
2	$\frac{161}{72}$	4×10^{-5}
3	$\frac{1}{2}\frac{161}{72} + \frac{5}{2}\frac{72}{161}$	10^{-10}

Notice that the number of digits of accuracy doubles with each iteration; x_2 is as good an approximation as you'll ever need, and x_3 is as good an approximation as the one displayed by your calculator.