Comparing Quadratic Approximations to Calculator Computations

In a previous worked example, we explored linear approximations to the sine function at the point $x = 0$. In this example, we use the quadratic approximation for e^x to calculate values of the exponential function near $x = 0$ and again compare the results to decimal approximations on a scientific calculator.

Find the quadratic approximation to e^x at the point $x = 0$ and use your answer to approximate the values of $e^{.01}, e^{.1}$ and e. Check your answer on a calculator.

Solution:

In the lecture, you learned that quadratic approximation to a function $f(x)$ at a point $x = a$ was given by a particular quadratic polynomial. This polynomial $Q(x)$ should be chosen so that $Q(a) = f(a), Q'(a) = f'(a)$ and $Q''(a) = f''(a)$. In the case where $f(x) = e^x$, we saw from lecture that the quadratic approximation at $x = 0$ was given by:

$$\begin{aligned} Q(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 &= e^0 + e^0(x-0) + \frac{e^0}{2}(x-0)^2 \\ &= 1 + x + \frac{1}{2}x^2. \end{aligned}$$

In short, we write $e^x \approx 1 + x + \frac{1}{2}x^2$ when $x \approx 0$. (It's very illuminating to again draw a picture of the exponential curve and its quadratic approximation at $x = 0$ to illustrate this.) So we would approximate the values of sine above as follows:

$$\begin{aligned} e^{.01} &\approx 1 + .01 + (.01)^2/2 = 1.01005 \\ e^{.1} &\approx 1 + .1 + (.1)^2/2 = 1.105 \\ e^1 &\approx 1 + 1 + 1^2/2 = 2.5 \end{aligned}$$

As in the previous worked example, we expect the approximations at values closest to $x = 0$ (where the quadratic approximation agrees with the function) will be the most accurate. The calculator confirms this.

$$\begin{aligned} e^{.01} &= 1.0100501670... \\ e^{.1} &= 1.1051709180... \\ e &= 2.7182818284... \end{aligned}$$

where we've only recorded the first ten digits of the decimal expansion. Notice that $e^{.01}$ only differs from our estimate 1.01005 by less than .00000017, an extremely accurate approximation! Yet e differs from 2.5 by more than .21. Again, the approximation will be poor for large values of x (i.e. far from $x = 0$) since a quadratic function grows much more slowly than an exponential function. In general, these approximations should only be relied upon for values near the point $x = a$ at which we perform the approximation. Much later in the course, we'll have a quantitative estimate for the error often referred to as "Taylor's theorem with remainder."