PROFESSOR: Hi. Welcome back to recitation.

In lecture you've been talking about implicitly defined functions and implicit differentiation. So one of the reasons that these are important is-- or that implicit differentiation is important, is that sometimes you have a function to find implicitly and you can't solve for it. You don't have any algebraic method for computing the function values as a formula, say.

So, for example, this function that I've written on the board that I've called w of x is defined implicitly by the equation that w of x plus 1 quantity times e to the w of x is equal to x for all x. So this function, some of its values you can guess. Like at x equals 0, the function value is going to be negative 1. And the reason is that this can't ever be 0, so the only way to get this side to be 0 is if w is negative 1 if this term is 0. So some of its values are easy to compute, but some of its values aren't.

So for example, if I asked you what w of 3/2 is, it's very hard. There's no algebraic way you can manipulate this equation that will let you figure that out. So in that situation you might still care about what the function value is. So what can you do? Well, you can try and find a numerical approximation. So in this problem I'd like you to try and estimate the value w of 3/2 by using a linear approximation to the function, to the curve-- yeah. A linear approximation of the function w of x in order to compute this value. So as a hint, I've given you--

so you're trying to compute w of 3/2. As a hint I'm pointing out to you that w of 1 is 0. Right? If you put in x equals 0 and w of 0 equals-- sorry-- if you put in x equals 1 and w of 1 equals 0 on the left hand side, you do indeed get 1, as you should. So, OK. So, good.

So that'll give you a hint about where you could base your linear approximation. So why don't you pause the video, take a few minutes to work this out, come back, and we can work it out together.

All right. Welcome back. So hopefully you've had a chance to work on this question a little bit.

So in order to do this linear approximation that we want, what we need to know are: we need to know a base point and we need to know the derivative of the function at that base point. And those are the two pieces of data you need in order to construct a linear approximation.

So we have a good candidate for a base point here, which is the point (1, 0). So this curve, whatever it looks like, it passes through the point (1, 0). And that's the point we're going to use for our approximation. So we're going to use the linear approximation w of x is approximately equal to w prime of 1 times x minus 1 plus w of 1 when x is approximately equal to 1.

So this is the linear approximation we're going to use, and we have that w of 1 here is 0. So this is, this is equal to w prime of one times x minus one. Just, the w of 1 is 0. It just goes away.

So in order to estimate w of x, and in particular w of 3/2, what we need to know is we need to know the derivative of w. OK? And to get the derivative of w, we need to use-- well, we have only one piece of information about w. Which is we have that it's defined by this implicit equation. So in order to get the derivative of w we have to use implicit differentiation. OK?

So let's do that. So if we implicitly differentiate this equation-- so let's start with the-- the right-hand side is going to be really easy. Right? We're going to differentiate with respect to x. The right-hand side is going to be 1. On the left-hand side it's going to be a little more complicated. We have a product and then this piece, we're going to have a chain rule situation. Right? We have e to the w of x. So, OK.

So we're going to take an implicit derivative and on the left-- so OK, so the product rule first. We take the derivative of the first part, so that's just w prime of x times the second part-- that's e to the w of x-- plus the first part-- that's w of x plus 1-- times the derivative of the second part. So the second part is e to the w of x. So that gives me an e to the w of x times w prime of x. That's the chain rule. So that's what happens when I differentiate the left-hand side. And on the right-hand side I take the derivative of x and I get 1. OK, good.

So now I've got this equation and I need to solve this equation for w prime. Because if you look up here, that's what I want. I want a particular value of w prime. And as always happens in implicit differentiation, from the point of view of this w prime it's only involved in the equation in a very simple way. So there's it multiplied by functions of x and w of x, but not-- you know, it's just multiplied by something that doesn't involve w prime at all. And then here it's multiplied by something that doesn't involve w prime at all. So you can just collect your w prime's and divide through. You know, it's just like solving a linear equation.

So here, if we collect our w prime's, this is w prime of x times-- looks like w of x plus 2 times e to the w of x. Did I get that right? Looks good. OK, so that's still equal to 1. So that means that w prime of x is just-- well, just, you know-- it's equal to 1 over w of x plus 2 times e to the w of x.

OK, so this is true for every x. But I don't need this equation for every x. I just need the particular value of w prime at 1. So that's, so I'm going to take this equation, then, and I'm just going to put in x equals 1. So I put in x equals 1-- well, let me do it over here-- so I get w prime of 1. And I just, everywhere I had an x, I put in a 1. So actually, in this equation the only place x appears is in the argument of w. So this is w of 1 plus 2 times e to the w of 1.

OK. So in order to get w prime of 1, I need to know what w of 1 is. But I had that. I had it, it was right back here. There was the-- that was my hint to you. Right, this is why we're using this point as a base point, which is we know the value of w for this value of x. So we take that value. So w of 1 is 0. So this is just 1 over-- well, 0 plus 2 is 2, and e to the 0 is 1. So it's just 1 over 2. Sorry, that's written a little oddly. We can make it 2 times 1. So 1 over 2.

OK. So I take that back upstairs to this equation that I had here. And I have that w of x is approximately equal to w prime of 1 times x minus 1. So w of x is approximately equal to-- w prime of 1, we saw is 1/2-- times x minus 1. And that approximation was good near our base point. So that's good when x is near 1.

All right. And then-- so this the linear approximation. And I asked for the linear approximation, its value at the particular point x equals 3/2. So w of 3/2 is approximately 1/2 times-- well, 3/2 minus 1 is also 1/2-- so this is a quarter. OK, so this is our estimate for w of 3/2. w of 3/2 is approximately 1/4.

If you wanted a better estimate you could try iterating this process. Now you might have a, you know-- or choosing some base point even closer if you could figure out the value of w and x near that, near this point that you're interested in, 3/2.

So just to sum up what we did was we had this implicitly defined function w. We wanted to estimate its value at a point where we couldn't compute it explicitly. So what we did was we did our normal linear approximation method. Right? So we wrote down our normal linear approximation formula. The only thing that was slightly unusual is that we had to use implicit differentiation. In order to compute the derivative that appears in the linear approximation, we implicitly differentiated. OK? So that happened just like normal, and then at the end we plugged in the values that we were interested in, to actually compute the particular value of that approximation.

So I'll end there.