The basic theorems on power series.

Whenever we have a series $\sum \mu_n(x)$ of functions, there are three fundamental questions we ask:

(1) Given the series $\sum \mu_n(x)$, for what values of x does the series converge?

(2) Given $\sum \mu_n(x)$, if it converges to a function $f(x)$, what properties does $f(x)$ have? Specifically: Is f continuous? Can you calculate $\int_a^b f(x)$ by integrating the series term-by-term? Is f differentiable, and can you calculate $f'(x)$ by differentiating the series term-by-term?

(3) Given a function $f(x)$, under what conditions does it equal such a series, where the functions $\mu_n(x)$ are functions of a particular type?

We shall answer these questions for a power series. This is a series of the form:

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n.$$

Theorem 1. Given a power series $\sum a_nx^n$, exactly one of the following holds:

(a) The series converges only for $x = 0$.

(b) The series converges absolutely for all x.

(c) There is a number $r > 0$ such that the series converges absolutely if $|x| < r$ and diverges if $|x| > r$. (Nothing is said about what happens when $x = \pm r$.)

Proof. Step 1. We show that if the series $\sum a_nx^n$ converges for $x = x_0 \neq 0$, then it converges (absolutely) for any x with $|x| < |x_0|$.

For this purpose, we write

$$|a_n x^n| = |a_n x_0^n|\ |x/x_0|^n = c_n |x/x_0|^n ,$$

where $c_n = |a_n x_0^n|$. Now the series $\sum |x/x_0|^n$ converges, because it is a geometric series of the form $\sum y^n$ with $|y| < 1$. Furthermore, the sequence c_n approaches 0 as $n \to \infty$, because the series $\sum a_n x_0^n$ converges (by hypothesis). We can choose N so that $|a_n x_0^n| \leq 1$ for $n \geq N$. Then $|a_n x^n| \leq |x/x_0|^n$ for $n \geq N$. The comparison thest then implies that the series $\sum |a_n x^n|$ converges.

<u>Step 2</u>. Let S be the set of all numbers x for which the series $\sum a_n x^n$ converges. If S consists of 0 alone, then (a) holds. Otherwise, there is at least one number x_0 different from 0 belonging to S. It then follows that there is a <u>positive</u> number x_1 belonging to S; indeed, if x_1 is any positive number such that $x_1 < |x_0|$, then $\sum |a_n x_1^n|$ converges by Step 1, so that $\sum a_n x_1^n$ converges and x_1 belongs to S. We now consider two cases.

<u>Case 1</u>. The set S is bounded above. In this case, we set $r = \sup S$, and show that the series $\sum a_n x^n$ diverges if $|x| > r$ and converges (absolutely) if $|x| < r$.

Divergence if $|x| > r$ is clear. For suppose $|x| > r$ and the series $\sum a_n x^n$ converges. If we choose x_2 so that $r < x_2 < |x|$, then Step 1 implies that the series $\sum |a_n x_2^n|$ converges. Then the series $\sum a_n x_2^n$ converges, so that x_2 belongs to S, contradicting the fact that r is an upper bound for S.

Convergence if $|x| < r$ is also clear. If $|x| < r$, we can choose an element x_3 of S such that $|x| < x_3$ (otherwise $|x|$ would be a smaller upper bound on S than r). Step 1 then implies that $\sum |a_n x^n|$ converges.

Case 2. The set S is unbounded above. We show the series $\sum a_n x_n$ converges (absolutely) for all x. Given x, choose an element x_4 of S such that $|x| < x_4$. This we can do because S is unbounded above. Then $\sum |a_n x^n|$ coverges, by Step 1. □

Definition. The number r constructed in (c) of the preceding theorem is called the radius of convergence of the series. In case (a) we say that $r = 0$; and in case (b), we say that $r = \infty$.

Theorem 2. *Suppose* $\sum a_n x^n$ *has radius of convergence* $r > 0$. (*We allow* $r = \infty$.) *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(a) f *is continuous for* $|x| < r$.

(b) *For* $|x| < r$, *we have*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}.$$

(c) *For* $|x| < r$, *we have*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

(d) *The series in* (b) *and* (c) *have radii of convergence precisely* r.

<u>Proof</u>. In general, let

$$p_m(x) = a_0 + a_1 x + \cdots + a_m x^m.$$

It is a polynomial of degree m, and it is the $m^{\underline{th}}$ partial sum of our power series.

We are going to prove parts (a) and (b) for the fixed point b in $(-r,r)$. So as a preliminary, let us choose R so that $|b| < R < r$.

<u>Step 1</u>. <u>Given</u> $\varepsilon > 0$, <u>there is an integer</u> N <u>such that the inequality</u>

$$|f(x) - p_m(x)| < \varepsilon$$

<u>holds for all</u> $m \geqslant N$ <u>and all</u> x <u>with</u> $|x| \leqslant R$.

The proof is easy. Since $\sum |a_n R^n|$ converges, we can choose N sufficiently large that

$$\sum_{n=N+1}^{\infty} |a_n R^n| < \varepsilon.$$

It follows that if $|x| \leqslant R$ and $m \geqslant N$, we have

$$\sum_{n=m+1}^{\infty} |a_n x^n| \leqslant \sum_{n=m+1}^{\infty} |a_n R^N| \leqslant \sum_{n=N+1}^{\infty} |a_n R^N| < \varepsilon.$$

Then for $m \geqslant N$ and $|x| \leqslant R$,

$$|f(x) - p_m(x)| = \left|\sum_{n=m+1}^{\infty} a_n x^n\right| \leqslant \sum_{n=m+1}^{\infty} |a_n x^n| < \varepsilon.$$

Step 2. We show that f is continuous at b. This proves (a).

Given $\varepsilon > 0$, choose N as in Step 1. We have

$$|f(x) - p_N(x)| < \varepsilon,$$

for any x in the interval $[-R,R]$. In particular,

$$|f(b) - p_N(b)| < \varepsilon.$$

Now we use continuity of the polynomial $p_N(x)$ to choose δ so that whenever $|x-b| < \delta$, then x is in $[-R,R]$, and

$$|p_N(x) - p_N(b)| < \varepsilon.$$

Adding these three inequalities and using the triangle inequality, we see that whenever $|x-b| < \delta$, we have

$$|f(x) - f(b)| < 3\varepsilon.$$

Step 3. We show that $\sum a_n b^{n+1}/(n+1)$ converges to $\int_0^b f(x)\, dx$. This proves (b).

Given $\varepsilon > 0$, choose N as in Step 1. Then whenever $m \geqslant N$, the inequality

$$-\varepsilon < f(x) - p_m(x) < \varepsilon$$

holds <u>for all</u> x in the interval $[-R,R]$. The comparison property of integrals tells us that

$$\left|\int_0^b (f(x)-p_m(x))\,dx\right| \leqslant \varepsilon|b|.$$

This says that

$$\left|\int_0^b f(x)\,dx - (a_0b+a_1b^2/2+\cdots+a_mb^{m+1}/(m+1))\right| \leqslant \varepsilon|b|$$

for all $m \geqslant N$. It follows that $\sum a_nb^{n+1}/(n+1)$ converges to $\int_0^b f(x)\,dx$.

<u>Step 4</u>. We show that the power series

$$\sum_{n=1}^{\infty} na_nx^{n-1}$$

has radius of convergence at least r.

For this purpose, it suffices to show that if c is any number with $0 < c < r$, then $\sum na_nc^{n-1}$ converges. In fact, it suffices to show that $\sum na_nc^n$ converges, since multiplying the series by c does not affect convergence. This is what we shall show.

First, choose d such that $c < d < r$. Then write the general term of our series in the form

$$na_nc^n = na_n(\tfrac{c}{d})^n \cdot d^n .$$

We note that the series $\sum a_nd^n$ converges because $d < r$. It follows that the n^{th} term a_nd^n approaches 0 as n becomes large. Choose N

sufficiently large that $|a_n d^n| < 1$ for $n \geq N$. Then for $n \geq N$, we have

$$na_n c^n \leq n(\frac{c}{d})^n .$$

Now the series

$$\sum n(\frac{c}{d})^n$$

converges by the ratio test, since $0 < c/d < 1$. Therefore the series $\sum na_n c^n$ converges, by the comparison test.

Step 5. We prove part (c). Let

$$g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

for $|x| < r$. Part (b) of the theorem tells us that for $|x| < r$, we have

$$\int_0^x g(t)\, dt = \sum_{n=1}^{\infty} a_n x^n.$$

$$= f(x) - a_0.$$

Part (a) of the theorem tells us that $g(x)$ is continuous for $|x| < r$. Then the first fundamental theorem of calculus applies; we conclude that

$$g(x) = f'(x),$$

which is what we wanted to prove.

<u>Step 6</u>. We prove part (d). If the series $\sum a_n x^{n+1}/(n+1)$ had radius of convergence $q > r$, then so would the differentiated series $\sum a_n x^n$, by Step 4. But it does not. Similarly, if the series $\sum na_n x^{n-1}$ had radius of convergence $q > r$, then so would the integrated series $\sum a_n x^n$. But it does not. $\square$

<u>Remark</u>. It follows readily that all the results of Theorem 2 hold for general power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

There is a number r (which may be 0 or ∞) such that the series converges absolutely for $|x-a| < r$ and diverges for $|x-a| > r$. Furthermore for $|x-a| < r$, one has:

(a) $f(x)$ is continuous.

(b) $\displaystyle\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$.

(c) $\displaystyle f'(x) = \sum_{n=1}^{\infty} na_n (x-a)^{n-1}$.

The proof is immediate; one merely substitutes $(x-a)$ for x in the theorem.

Here is a theorem proving the uniqueness of a power series representation:

<u>Theorem 3</u>. <u>Suppose</u>

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} b_n (x-a)^n$$

<u>on some open interval</u> I <u>containing</u> a. <u>Then for all</u> k,

$$a_k = b_k = \frac{f^{(k)}(a)}{k!}.$$

Proof. We apply the preceding theorem. We write

$$f(x) = \sum a_n (x-a)^n.$$

Differentiating, we have

$$f'(x) = \sum_{n=1}^{\infty} na_n (x-a)^{n-1}.$$

Applying the theorem once again, we have

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n (x-a)^{n-2}.$$

And so on. Differentiating k times, we have

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k}.$$

When we evaluate at $x = 0$, all the terms vanish except for the first term. Thus

$$f^{(k)}(x) = k!a_k,$$

as desired. The same argument applies to compute b_k. $\square$

Definition. If $f(x)$ equals a power series $\sum a_n (x-a)^n$ in some open interval containing a, we say f is analytic (or sometimes "real analytic") near a. By the preceding theorem, this power series is uniquely determined by f; its partial sums must be the Taylor polynomials of f at a. For this reason, the series is sometimes called the Taylor series of f at a.

Corollary 4. The function $f(x)$ is analytic near a if and only if both the following hold:

(1) All derivatives of f exist in an open interval about a.

(2) The error term $E_n(x)$ in Taylor's formula approaches 0 as $n \longrightarrow \infty$, for each x in that interval.

Remark. We know that it is possible for us to have

$$f(x) = \sum_{n=0}^{\infty} \mu_n(x)$$

for all x in an interval $[c,d]$, where each function $\mu_n(x)$ is continuous, without it following that $f(x)$ is continuous, or that its integral can be obtained by integrating the series term-by-term. However, this unpleasant situation does not occur if the analogue of the statement in Step 1 of the proof of Theorem 2 holds. This fact leads to the following definition.

Definition. The series $\sum \mu_n(x)$ is said to converge uniformly to $f(x)$ on the interval $[c,d]$ if given $\varepsilon > 0$, there is an N such that

$$\left| f(x) - \sum_{n=0}^{m} \mu_n(x) \right| < \varepsilon$$

for all $m \geq N$ and all x in $[c,d]$.

<u>Theorem 5</u>. <u>The series</u> $\sum \mu_n(x)$ <u>converges uniformly on</u> $[c,d]$ <u>if there is a convergent series</u> $\sum M_n$ <u>of constants such that</u> $|\mu_n(x)| < M_n$ <u>for all</u> x <u>in</u> $[c,d]$.

(The proof is just like that of Step 1. There the series of constants was the series $\sum |a_n R^n|$.)

Under the hypothesis of uniform convergence, the analogues of (a) and (b) of Theorem 2 hold:

<u>Theorem 6</u>. <u>Suppose</u> $\sum \mu_n(x)$ <u>converges uniformly to</u> $f(x)$ <u>on</u> $[c,d]$. <u>If the functions</u> $\mu_n(x)$ <u>are continuous, so is</u> $f(x)$, <u>and furthermore the series</u>

$$\sum_{n=0}^{\infty} \left(\int_c^x \mu_n(t)\,dt\right)$$

<u>converges uniformly to</u> $\int_c^x f(t)\,dt$ <u>on</u> $[c,d]$.

The proof is just like the ones given in Steps 2 and 3.

<u>Remark</u>. Part (c) of the theorem, about differentiating a power series term-by-term, does <u>not</u> carry over to more general uniformly convergent series. For instance, the series

$$\sum_{n=1}^{\infty} (\sin nx)/n^2$$

converges uniformly on any interval, by comparison with the series of constants $\sum 1/n^2$, but the differentiated series $\sum (\cos nx)/n$ does not even <u>converge</u> at $x = 0$. If however the differentiated series <u>does</u> converge uniformly on $[c,d]$, then $f'(x)$ does exist and equals this differentiated series. The proof is similar to that of (c).

Exercises

1. Prove Theorem 3.

2. Prove Theorem 4.

3. Prove the following theorem about term-by-term differentiation:

Suppose that the functions $\mu_n'(x)$ are continuous, that the series $\sum_{n=1}^{\infty} \mu_n'(x)$ converges uniformly on $[c,d]$, and that $\sum_{n=1}^{\infty} \mu_n(x)$ converges for at least one x in $[c,d]$. Then:

(a) $\sum_{n=1}^{\infty} \mu_n(x)$ converges uniformly on $[c,d]$, say to $f(x)$.

(b) $f'(x)$ exists and equals $\sum \mu_n'(x)$.

[Hint: Integrate the series $\sum \mu_n'(x)$.]