Theorem. Let m *and* n *be integers; let* $n > 0$. *Let*

$$h(y) = (\sqrt[n]{y})^m \quad \textit{for} \quad y > 0.$$

Then h *is differentiable, and*

$$h'(y) = \frac{m}{n} (\sqrt[n]{y})^m \cdot y^{-1} .$$

Proof. Step 1. We first prove the theorem in the case $m = 1$. Let $f(x) = x^n$ for $x > 0$. Then the inverse function to f, denoted $g(y)$, is the n^{th} root function. By the theorem on the derivative of an inverse function, $g'(y)$ exists and

$$g'(y) = \frac{1}{f'(x)} ,$$

where $x = g(y)$. Now $f'(x) = n\,x^{n-1}$. Therefore

$$\begin{aligned} g'(y) &= \frac{1}{n\,x^{n-1}} = \frac{1}{n(\sqrt[n]{y})^{n-1}} \\ &= \frac{1}{n(\sqrt[n]{y})^n\,(\sqrt[n]{y})^{-1}} = \frac{\sqrt[n]{y}}{n\,y} \\ &= \frac{1}{n}\sqrt[n]{y} \cdot y^{-1} . \end{aligned}$$

Step 2. We prove the theorem in general. If $m = 0$, it is trivial. Otherwise, we apply the chain rule. We have

$$h(y) = (\sqrt[n]{y})^m ;$$

then

$$\begin{aligned} h'(y) &= m(\sqrt[n]{y})^{m-1} \left[\frac{1}{n}\sqrt[n]{y} \cdot y^{-1}\right] \\ &= \frac{m}{n} (\sqrt[n]{y})^m \cdot y^{-1}. \quad \square \end{aligned}$$

Once one has checked that the laws of exponents hold for rational exponents (Notes G), one can write this formula in a manner that is much easier to remember:

Theorem. *Let* r *be a rational constant; let* $h(x) = x^r$ *for* $x > 0$. *Then* h *is differentiable and*

$$h'(x) = rx^{r-1}.$$

We will give a different proof of this theorem later on, one which holds when r is an arbitrary real constant.