18.01 Single Variable Calculus
Fall 2006

18.01 REVIEW PROBLEMS AND SOLUTIONS

Unit I: Differentiation

R1-0 Evaluate the derivatives. Assume all letters represent constants, except for the independent and dependent variables occurring in the derivative.

a) $pV^\gamma = nRT, \quad \dfrac{dp}{dV} = ?$ b) $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}}, \quad \dfrac{dm}{dv} = ?$

c) $R = \dfrac{a\omega_0 \sin(2k+1)\alpha}{\alpha^2+\beta^2}, \quad \left.\dfrac{dR}{d\alpha}\right|_{\alpha=0} = ?$

R1-1 Differentiate:

a) $\dfrac{\sin x}{x+1}$ b) $\sin^2(\sqrt{x})$ c) $x^{1/3}\tan x$

d) $\dfrac{x^2+2}{\sqrt{x+1}}$ e) $\cos(\sqrt{x^2+1})$ f) $\cos^3(\sqrt{x^2+1})$

g) $\tan(x^3)$ h) $x\sec^2(3x+1)$

R1-2 Consider $f(x) = 2x^2+4x+3$. Where does the tangent line to the graph of $f(x)$ at $x=3$ cross the y-axis?

R1-3 Find the equation of the tangent to the curve $2x^2+xy-y^2+2x-3y=20$ at the point (3,2).

R1-4 Define the derivative of $f(x)$. Directly from the definition, show that $f'(x) = \cos x$ if $f(x) = \sin x$. (You may use without proof: $\lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$, $\lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} = 0$).

R1-5 Find all real x_0 such that $f'(x_0) = 0$:

a) $f(x) = \dfrac{x}{x^2+1}$ b) $f(x) = x^2+\cos x$

R1-6 At what points is the tangent to the curve $y^2+xy+x^2-3=0$ horizontal?

R1-7 State and prove the formula for $(uv)'$ in terms of the derivatives of u and v. You may assume any theorems about limits that you need.

R1-8 Derive a formula for $(x^{1/5})'$.

R1-9 a) What is the rate of change of the area A of a square with respect to its side x ?

b) What is the rate of change of the area A of a circle with respect to its radius r?

c) Explain why one answer is the perimeter of the figure but the other answer is not.

R1-10 Find all values of the constants c and d for which the function $f(x) = \begin{cases} x^2+1, & x \geq 1 \\ cx+d, & x < 1 \end{cases}$ will be (a) continuous, (b) differentiable.

R1-11 Prove or give a counterexample :

a) If $f(x)$ is differentiable then $f(x)$ is continuous.

b) If $f(x)$ is continuous then $f(x)$ is differentiable.

R1-12 Find all values of the constants a and b so that the function $f(x) = \begin{cases} \sin x, & x \leq \pi \\ ax+b, & x > \pi \end{cases}$ will be (a) continuous; (b) differentiable.

R1-13 Evaluate $\lim_{x \to 0} \frac{\sin(4x)}{x}$. (Hint: Let $4x = t$.)

Unit 2: Applications of Differentiation

R2-1 Sketch the graphs of the following functions, indicating maxima, minima, points of inflection, and concavity.

a) $f(x) = (x-1)^2(x+2)$

b) $f(x) = \sin^2 x, \quad 0 \leq x \leq 2\pi$

c) $f(x) = x + 1/x^2$

d) $f(x) = x + \sin 2x$

R2-2 A baseball diamond is a 90 ft. square. A ball is batted along the third base line at a constant speed of 100 ft. per sec. How fast is its distance from first base changing when

a) it is halfway to third base,

b) it reaches third base ?

R2-3 If x and y are the legs of a right triangle whose hypotenuse is $\sqrt{5}$, find the largest value of $2x + y$.

R2-4 Evaluate the following limits:

a) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$

b) $\lim_{x \to 0} \frac{\sin x}{x}$

c) $\lim_{x \to \infty} \frac{x^{17} - 4x^3 + 2x^2}{10x^{17} + 6x^{10} - x^3 - 5x^2}$

R2-5 Prove or give a counterexample:

a) If $f'(c) = 0$ then f has a minimum or a maximum at c.

b) If f has a maximum at c and if f is differentiable at c, then $f'(c) = 0$.

R2-6 Let $f(x) = 1 - x^{2/3}$. Then $f(-1) = f(1) = 0$ and yet $f'(x) \neq 0$ for $0 < x < 1$. Find the maximum value of $f(x)$ on the real line, nevertheless.

Why did the standard method fail?

R2-7 A can is made in the shape of a right circular cylinder. What should its proportions be, if its volume is to be 1 and one wants to use the least amount of metal?

R2-8 a) State the mean value theorem.

b) If $f'(x) = \dfrac{1}{1+x^2}$ and $f(1) = 1$, use the mean value theorem to estimate $f(2)$. (Write your answer in the form $\alpha < f(2) < \beta$.)

R2-9 One of these statements is false and one is true. Prove the true one, and give a counterexample to the false one. (Both statements refer to all x in some interval $a < x < b$.)

a) If $f'(x) > 0$, then $f(x)$ is an increasing function.

b) If $f(x)$ is an increasing function , then $f'(x) > 0$.

R2-10 Give examples (either by giving a formula or by a carefully drawn graph) of

a) A function with a relative minimum, but no absolute maximum on $0 < x < 1$.

b) A function with a relative maximum but no absolute maximum on the interval $0 \le x \le 1$.

c) A function $f(x)$ defined on $0 \le x \le 1$, with $f(0) < 0$, $f(1) > 0$, yet with no root on $0 \le x \le 1$.

d) A function $f(x)$ having a relative minimum at 0, but the following is false: $f'(0) = 0$.

Unit 3: Integration

R3-1 Evaluate: $\displaystyle\int_0^{\pi} \sin x\,dx, \int_0^3 \sqrt{1+x}\,dx, \int_1^2 \frac{x^2+1}{x^2}\,dx.$

R3-2 Egbert, an MIT nerd bicyclist, is going down a steep hill. At time $t = 0$, he starts from rest at the top of the hill; his acceleration while going down is $3t^2$ ft./sec^2, and the hill is 64 ft. long. If the fastest he can go without losing control is 64 ft./sec., will he survive this harrowing experience? (A nerd bicycle has no brakes.)

R3-3 Evaluate $\int_0^2 x^2 dx$ directly from the definition of the integral as the limit of a sum. You may use the fact that

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

R3-4 If f is a continuous function, find $f(2)$ if:

a) $\displaystyle\int_0^x f(t)dt = x^2(1+x)$ b) $\displaystyle\int_0^{x^2} f(t)dt = x^2(1+x)$ c) $\displaystyle\int_0^{f(x)} t^2 dt = x^2(1+x)$

R3-5 The area under the graph of $f(x)$ and over the interval $0 \le x \le a$ is

$$-\frac{1}{2} + \frac{a^2}{4} + \frac{a}{2}\sin a + \frac{1}{2}\cos a$$

Find $f(\pi/2)$.

R3-6 Use the trapezoidal rule to estimate the sum $\sqrt[3]{1}+\sqrt[3]{2}+\cdots+\sqrt[3]{10^6}$. Is your estimate high or low? Explain your reasoning.

R3- 7 Find the total area of the region above the graph of $y=-2x$ and below the graph of $y=x-x^2$.

R3-8 Use the trapezoidal rule with 6 subintervals to estimate the area under the curve $y=\sqrt{1+x^2}, -3\leq x\leq 3$. (You may use: $\sqrt{2}\approx 1.41$, $\sqrt{5}\approx 2.24$, $\sqrt{10}\approx 3.16$.

Is your estimate too high or too low? Explain how you know.)

R3-9 Fill in this outline of a proof that $\int_a^b F'(x)dx=F(b)-F(a)$. Supply reasons.

a) Put $G(x)=\int_a^x F'(t)dt$. Then $G'(x)=F'(x)$.

b) Therefore $G(x)=F(x)+c$, and one sees easily that $c=-F(a)$. We're done.

R3-10 The table below gives the known values of a function $f(x)$:

x	0	1	2	3	4	5	6
$f(x)$	1	1.2	1.4	1.3	1.5	1.2	1.1

Use Simpson's Rule to estimate the area under the curve $y=f(x)$ between $x=0$ and $x=6$.

R3-11 Let $f(t)$ be a function, continuous and positive for all t. Let $A(x)$ be the area under the graph of f, between $t=0$ and $t=x$. Explain intuitively from the definition of derivative why $\frac{dA}{dx}=f(x)$.

R3-12 Let $f(x)=\begin{cases} x+1, & 0\leq x\leq 2\\ x-2, & 2<x\leq 4\end{cases}$ Evaluate $\int_0^4 f(x)dx$.

R3-13 Suppose $F(x)$ is a function such that $F'(x)=\frac{\sin x}{x}$. In terms of $F(x)$, evaluate the indefinite integral $\int \frac{\sin 3x}{x}dx$ ..

R3-14 Find a quadratic polynomial $ax^2+bx+c=f(x)$ such that $f(0)=0$, $f(1)=1$, and the area under the graph between $x=0$ and $x=1$ is 1.

Unit 4: Applications of integration.

R4-1 The area in the first quadrant bounded by the lines $y = 1, x = 1, x = 3$ and $f(x) = -x^2 + 15$ is rotated about the line $y = 1$. Find the volume of the solid thus obtained.

R4-2 Consider the circle $x^2 + y^2 = 4$. A solid is formed with the given circle as base and such that every cross-section cut by a plane perpendicular to the x-axis is a square. Find the volume of this solid.

R4-3 Find the length of the arc of $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

R4-4 For a freely falling body, $s = \frac{1}{2}gt^2, v = gt = \sqrt{2gs}$. Show that:

a) the average value of v over the interval $0 \le t \le t_1$ is one-half the final velocity;

b) the average value of v over the interval $0 \le s \le s_1$ is two-thirds the final velocity.

R4-5 A bag of sand originally weighing 144 pounds is lifted at a constant rate of 3 ft./min. the sand leaks out uniformly at such a rate that half the sand is lost when the bag has been lifted 18 feet. find the work done in lifting the bag this distance.

R4-6 Find the area inside both loops of the lemniscate $r^2 = 2a^2 \cos 2\theta$.

R4-7 Calculate the volume obtained when the region $(-2 \le x \le 2, 0 \le y \le x^2)$ is rotated about the y-axis.

R4-8 The table below gives the known values of a function $f(x)$:

x	0	1	2	3	4	5	6
$f(x)$	1	1.2	1.4	1.3	1.5	1.2	1.1

Use Simpson's Rule to estimate the volume obtained when the region below the graph of $y = f(x)$ and above the x-axis $(0 \le x \le 6)$ is rotated about the x-axis.

R4-9 Winnie the Pooh eats honey at a rate proportional to the amount he has left. If it takes him 1 hour to eat the first half of a pot of honey, how long will it take for him to eat another quarter of a pot? When will he finish?

R4-10 a) Write down the definition of $\ln x$ as an integral.

b) Directly from the definition prove that:

i) $\ln(ax) = \ln a + \ln x$; ii) $\ln x$ is an increasing function.

Unit 5: Integration Techniques

R5-1 Differentiate:

a) $x^{1/x}$, $e^{x^2} \cdot \ln(x^2)$ b) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$.

R5-2 Integrate:

a) $\int \sin^3 x \cos^2 x dx$ b) $\int e^x \sin x dx$

R5-3 Integrate:

a) $\int \frac{e^x}{1+e^{2x}} dx$ b) $\int \frac{x+1}{x^3-1} dx$ c) $\int \frac{4x^2}{x-2} dx$

R5-4 Integrate:

a) $\int \frac{x+1}{(1+x^2)^2} dx$ b) $\int x^2 \cos x dx$

R5-5 a) Use the reduction formula

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

to evaluate $\int_0^{\pi/2} \cos^6 x dx$.

b) Derive the formula for $D \tan^{-1} x$ from the formula for $D \tan x$. What are the domain and range of $\tan^{-1} x$?

SOLUTIONS TO 18.01 REVIEW PROBLEMS

Unit 1: Differentiation

R1-0.

a) $\dfrac{-\gamma nRT}{V^{\gamma+1}}$ b) $\dfrac{m_0 v}{c^2(1-v^2/c^2)^{3/2}}$ c) $\dfrac{c\omega_0(2k+1)}{\beta^2}$

R1-1

a) $\dfrac{(x+1)\cos x - \sin x}{(x+1)^2}$ b) $\dfrac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}$ c) $x^{1/3}\sec^2 x + \dfrac{1}{3}x^{-2/3}\tan x$

d) $\dfrac{\frac{3x^2}{2}+2x-1}{(\sqrt{x}+1)^3}$ e) $\dfrac{-\sin\sqrt{x^2+1}\times x}{\sqrt{x^2+1}}$ f) $\dfrac{(3\cos^2\sqrt{x^2+1})(-\sin\sqrt{x^2+1})x}{\sqrt{x^2+1}}$

g) $3x^2\sec^2(x^3)$ h) $\sec^2(3x+1)+6x\sec^2(3x+1)\tan(3x+1)$

R1-2 $(0,-15)$

R1-3 $y = 4x - 10$

R1-4 Hint: Use addition formula to expand $\sin(x+\Delta x)$.

R1-5 a) $x = \pm 1$. b) $x = 0$

R1-6 $(1,-2)$ and $(-1,2)$

R1-7 See Simmons, sec. 3.2

R1-8 Hint: Differentiate implicitly the equation $y^5 = x$.

R1-10 a) $c+d=2$ b) $c=2, d=0$

R1-11 a) cf. p. 75, Simmons. b) false; $f(x)=|x|$

R1-12 a) $b=-a\pi$ b) $a=-1, b=\pi$

R1-13 Let $4x=t$. $\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{x} = \lim_{\frac{t}{4}\to 0}\frac{4\sin(t)}{t} = 4\lim_{t\to 0}\frac{\sin(t)}{t} = 4$

Unit 2: Applications of Differentiation

R2-2 a) $20\sqrt{5}$ ft/sec b) $50\sqrt{2}$ ft/sec

R2-3 5 **R2-4** a) 1 b) 1 c) $\dfrac{1}{10}$

R2-5 a) false b) cf. p. 801—802 (1), (2), (3) Simmons. **R2-6** 1

R2-7 $r=(2\pi)^{-1/3}$, $h=(4/\pi)^{1/3}$

R2-8 b) $\frac{6}{5} < f(2) < \frac{3}{2}$

R2-9 a) This is true, use mean value theorem. b) This is false ; try x^3.

R2-10 a) see graph b) $f(x)$ must be discontinuous c) $f(x)$ is discontinuous d) $|x|$

Unit 3: Integration

R3-1 2, $\frac{14}{3}, \frac{3}{2}$

R3-2 Yes. Hint: Find the time it takes him to reach the bottom of the hill, and find his speed at that instant.

R3-3 $\frac{8}{3}$ **R3-4** a) 16 b) $1 + \frac{3\sqrt{2}}{2}$ c) $(36)^{1/3}$

R3-5 $\frac{\pi}{4}$ **R3-6** $\leq 75,000,050$ **R3-7** $\frac{9}{2}$ **R3-8** 11.46

R3-10 7.566... **R3-12** 6 **R3-14** $4x - 3x^2$

Unit 4: Applications of Integration

R4-1. $\pi \times 197\frac{11}{15}$ **R4-2.** $\frac{128}{3}$ **R4-3.** 12

R4-5. 1944 ft.lbs **R4-6.** $2a^2$ **R4-7.** 8π

R4-8. 8. $\pi \times 9.636...$ **R4-9.** Another hour; never.

R4-10 b) hint: write $\int_1^{ab} f(t)dt = \int_1^a f(t)dt + \int_a^{ab} f(t)dt$

Unit 5: Integration Techniques

R5-1 $x^{1/x}(\frac{1-\ln x}{x^2}), e^{x^2}(\frac{2}{x} + 2x\ln(x^2)), \frac{1}{1+x^2}$

R5-2 a) $\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c$ b) $\frac{e^x}{2}(\sin x - \cos x) + c$

R5-3 a) $\tan^{-1}(e^x) + c$ b) $\frac{1}{3}\ln\frac{(x-1)^2}{|x^2+x+1|} + c$ c) $2x^2 + 8x + 16\ln(x-2) + c$

R5-4 a) $\frac{1}{2}(\tan^{-1} x + \frac{x-1}{1+x^2}) + c$ b) $x^2 \sin x + 2x\cos x - 2\sin x + c$

R5-5 $\frac{5\pi}{32}$