18.01 Single Variable Calculus
Fall 2006

Lecture 34: Indeterminate Forms - L'Hôpital's Rule

L'Hôpital's Rule

(Two correct spellings: "L'Hôpital" and "L'Hospital")

Sometimes, we run into indeterminate forms. These are things like

$$\frac{0}{0}$$

and

$$\frac{\infty}{\infty}$$

For instance, how do you deal with the following?

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1} = \frac{0}{0}\,??$$

Example 0. One way of dealing with this is to use algebra to simplify things:

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x\to 1}\frac{x^2+x+1}{x+1} = \frac{3}{2}$$

In general, when $f(a) = g(a) = 0$,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\dfrac{f(x)}{x-a}}{\dfrac{g(x)}{x-a}} = \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

This is the easy version of L'Hôpital's rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Note: this only works when $g'(a) \neq 0$!

In example 0,

$$f(x) = x^3 = 1;\ g(x) = x^2 - 1$$

$$f'(x) = 3x^2;\ g'(x) = 2x \implies f'(1) = 3;\ g'(1) = 2$$

The limit is $f'(1)/g'(1) = 3/2$. Now, let's go on to the full L'Hôpital rule.

Example 1. Apply L'Hôpital's rule (a.k.a. "L'Hop") to

$$\lim_{x \to 1} \frac{x^{15} - 1}{x^3 - 1}$$

to get

$$\lim_{x \to 1} \frac{x^{15} - 1}{x^3 - 1} = \lim_{x \to 1} \frac{15x^{14}}{3x^2} = \frac{15}{3} = 5$$

Let's compare this with the answer we'd get if we used linear approximation techniques, instead of L'Hôpital's rule:

$$x^{15} - 1 \approx 15(x - 1)$$

(Here, $f(x) = x^{15} - 1, a = 1, f(a) = b = 0, m = f'(1) = 15$, and $f(x) \approx m(x - a) + b$.)
Similarly,

$$x^3 - 1 \approx 3(x - 1)$$

Therefore,

$$\frac{x^{15} - 1}{x^3 - 1} \approx \frac{15(x - 1)}{3(x - 1)} = 5$$

Example 2. Apply L'Hop to

$$\lim_{x \to 0} \frac{\sin 3x}{x}$$

to get

$$\lim_{x \to 0} \frac{3 \cos 3x}{1} = 3$$

This is the same as

$$\frac{d}{dx} \sin(3x)\Big|_{x=0} = 3\cos(3x)\Big|_{x=0} = 3$$

Example 3.

$$\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\cos x + \sin x}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$f(x) = \sin x - \cos x, \ f'(x) = \cos x + \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Remark: Derivatives $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ are always a $\frac{0}{0}$ type of limit.

Example 4. $\lim_{x \to 0} \frac{\cos x - 1}{x}$.
Use L'Hôpital's rule to evaluate the limit:

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{-\sin x}{x} = 0$$

Example 5. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

Just to check, let's compare that answer to the one we would get if we used quadratic approximation techniques. Remember that:

$$\cos x \approx 1 - \frac{1}{2}x^2 \quad (x \approx 0)$$

$$\frac{\cos x - 1}{x^2} \approx \frac{1 - \frac{1}{2}x^2 - 1}{x^2} = \frac{(-\frac{1}{2})x^2}{x^2} = -\frac{1}{2}$$

Example 6. $\lim_{x\to 0} \frac{\sin x}{x^2}$.

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} \quad \text{By L'Hôpital's rule}$$

If we apply L'Hôpital again, we get

$$\lim_{x\to 0} -\frac{\sin x}{2} = 0$$

But this doesn't agree with what we get from taking the linear approximation:

$$\frac{\sin x}{x^2} \approx \frac{x}{x^2} = \frac{1}{x} \to \infty \quad \text{as} \quad x \to 0^+$$

We can clear up this seeming paradox by noting that

$$\lim_{x\to 0} \frac{\cos x}{2x} = \frac{1}{0}$$

The limit is not of the form $\frac{0}{0}$, which means L'Hôpital's rule cannot be used. *The point is: look before you L'Hôp!*

More "interesting" cases that work.

It is also okay to use L'Hôpital's rule on limits of the form $\frac{\infty}{\infty}$, or if $x \to \infty$, or $x \to -\infty$. Let's apply this to rates of growth. Which function goes to ∞ faster: x, e^{ax}, or $\ln x$?

Example 7. For $a > 0$,

$$\lim_{x\to\infty} \frac{e^{ax}}{x} = \lim_{x\to\infty} \frac{ae^{ax}}{1} = +\infty$$

So e^{ax} grows faster than x (for $a > 0$).

Example 8.

$$\lim_{x\to\infty} \frac{e^{ax}}{x^{10}} = \text{by L'Hôpital} = \lim_{x\to\infty} \frac{ae^{ax}}{10x^9} = \lim_{x\to\infty} \frac{c^2 e^{ax}}{10\cdot 9x^8} = \cdots = \lim_{x\to\infty} \frac{a^{10}e^{ax}}{10!} = \infty$$

You can apply L'Hôpital's rule ten times. There's a better way, though:

$$\left(\frac{e^{ax}}{x^{10}}\right)^{1/10} = \frac{e^{ax/10}}{x}$$

$$\lim_{x\to\infty} \frac{e^{ax}}{x^{10}} = \lim_{x\to\infty} \left(\frac{e^{ax/10}}{x}\right)^{10} = \infty^{10} = \infty$$

Example 9.

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/3}} = \lim_{x\to\infty} \frac{1/x}{1/3x^{-2/3}} = \lim_{x\to\infty} 3x^{-1/3} = 0$$

Combining the preceding examples, $\ln x \ll x^{1/3} \ll x \ll x^{10} \ll e^{ax} \quad (x \to \infty, a > 0)$

L'Hôpital's rule applies to $\frac{0}{0}$ and $\frac{\infty}{\infty}$. But, we sometimes face other indeterminate limits, such as 1^∞, 0^0, and $0 \cdot \infty$. Use algebra, exponentials, and logarithms to put these in L'Hôpital form.

Example 10. $\lim_{x\to 0} x^x$ for $x > 0$.
Because the exponent is a variable, use base e:

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x \ln x}$$

First, we need to evaluate the limit of the exponent

$$\lim_{x\to 0} x \ln x$$

This limit has the form $0 \cdot \infty$. We want to put it in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Let's try to put it into the $\frac{0}{0}$ form:

$$\frac{x}{1/\ln x}$$

We don't know how to find $\lim_{x\to 0} \frac{1}{\ln x}$, though, so that approach isn't helpful.

Instead, let's try to put it into the $\frac{\infty}{\infty}$ form:

$$\frac{\ln x}{1/x}$$

Using L'Hôpital's rule, we find

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} (-x) = 0$$

Therefore,

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x \ln x} = e^{\lim_{x\to 0} (x \ln x)} = e^0 = 1$$