18.01 Single Variable Calculus
Fall 2006

Lecture 32: Polar Co-ordinates, Area in Polar Co-ordinates

Polar Coordinates

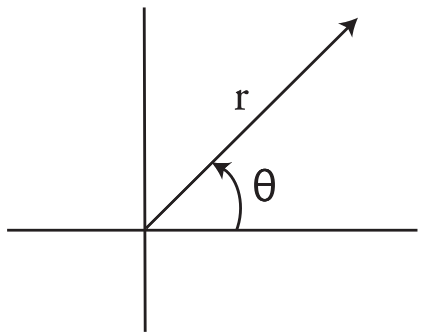

Figure 1: Polar Co-ordinates.

In polar coordinates, we specify an object's position in terms of its distance r from the origin and the angle θ that the ray from the origin to the point makes with respect to the x-axis.

Example 1. What are the polar coordinates for the point specified by $(1,-1)$ in rectangular coordinates?

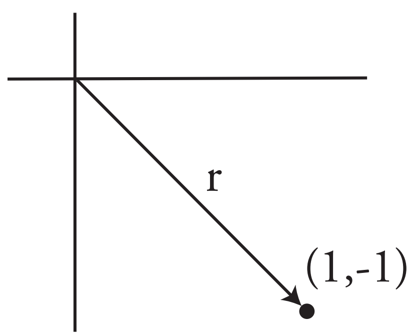

Figure 2: Rectangular Co-ordinates to Polar Co-ordinates.

$$\begin{aligned} r &= \sqrt{1^2+(-1)^2} = \sqrt{2} \\ \theta &= -\frac{\pi}{4} \end{aligned}$$

In most cases, we use the convention that $r \geq 0$ and $0 \leq \theta \leq 2\pi$. But another common convention is to say $r \geq 0$ and $-\pi \leq \theta \leq \pi$. All values of θ and even negative values of r can be used.

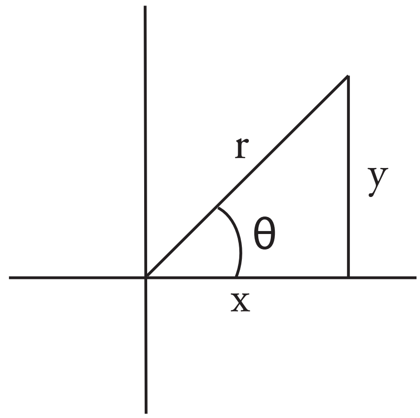

Figure 3: Rectangular Co-ordinates to Polar Co-ordinates.

Regardless of whether we allow positive or negative values of r or θ, what is *always* true is:

$$\boxed{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta}$$

For instance, $x = 1$, $y = -1$ can be represented by $r = -\sqrt{2}$, $\theta = \dfrac{3\pi}{4}$:

$$1 = x = -\sqrt{2}\cos\frac{3\pi}{4} \quad \text{and} \quad -1 = y = -\sqrt{2}\sin\frac{3\pi}{4}$$

Example 2. Consider a circle of radius a with its center at $x = a$, $y = 0$. We want to find an equation that relates r to θ.

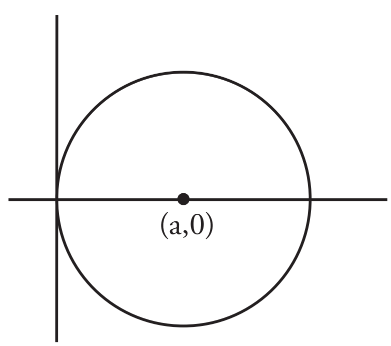

Figure 4: Circle of radius a with center at $x = a, y = 0$.

We know the equation for the circle in rectangular coordinates is

$$(x-a)^2 + y^2 = a^2$$

Start by plugging in:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

This gives us

$$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$$
$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$$
$$r^2 - 2ar\cos\theta = 0$$
$$\boxed{r = 2a\cos\theta}$$

The range of $0 \le \theta \le \frac{\pi}{2}$ traces out the top half of the circle, while $-\frac{\pi}{2} \le \theta \le 0$ traces out the bottom half. Let's graph this.

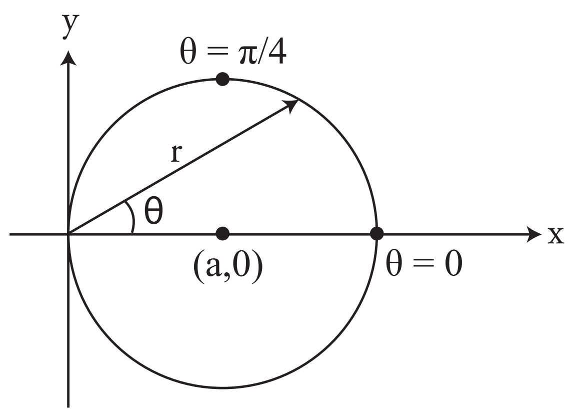

Figure 5: $r = 2a\cos\theta, \quad -\pi/2 \le \theta \le \pi/2.$

At $\theta = 0,\ r = 2a \implies x = 2a,\ y = 0$
At $\theta = \frac{\pi}{4},\ r = 2a\cos\frac{\pi}{4} = a\sqrt{2}$

The main issue is finding the range of θ tracing the circle once. In this case, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

$$\theta = -\frac{\pi}{2} \text{ (down)}$$
$$\theta = \frac{\pi}{2} \text{ (up)}$$

Weird range (avoid this one): $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. When $\theta = \pi$, $r = 2a\cos\pi = 2a(-1) = -2a$. The radius points "backwards". In the range $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, the same circle is traced out a second time.

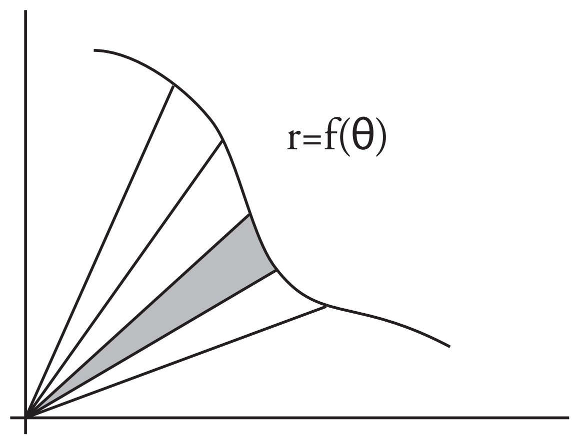

Figure 6: Using polar co-ordinates to find area of a generic function.

Area in Polar Coordinates

Since radius is a function of angle ($r = f(\theta)$), we will integrate with respect to θ. The question is: what, exactly, should we integrate?

$$\int_{\theta_1}^{\theta_2} ??\, d\theta$$

Let's look at a very small slice of this region:

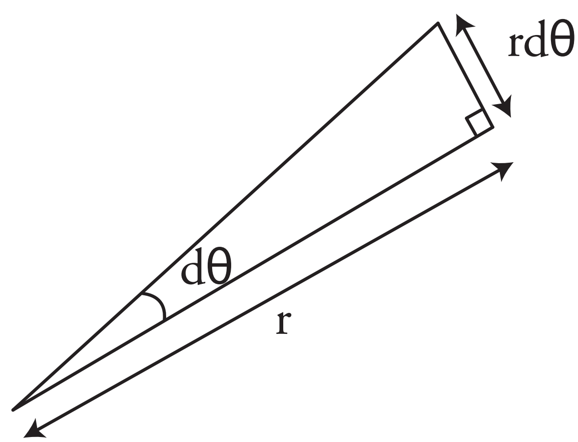

Figure 7: Approximate slice of area in polar coordinates.

This infinitesimal slice is approximately a right triangle. To find its area, we take:

$$\text{Area of slice} \approx \frac{1}{2}(\text{base})\,(\text{height}) = \frac{1}{2}r(r\,d\theta)$$

So,

$$\text{Total Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\, d\theta$$

Example 3. $r = 2a\cos\theta$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (the circle in Figure 5).

$$A = \text{area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(2a\cos\theta)^2\, d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

Because $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, we can rewrite this as

$$A = \text{area} = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta = a^2 \int_{-\pi/2}^{\pi/2} d\theta + a^2 \int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta$$

$$= \pi a^2 + \frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2} = \pi a^2 + \frac{1}{2}\underbrace{[\sin\pi - \sin(-\pi)]}_{0}$$

$$A = \text{area} = \pi a^2$$

Example 4: Circle centered at the Origin.

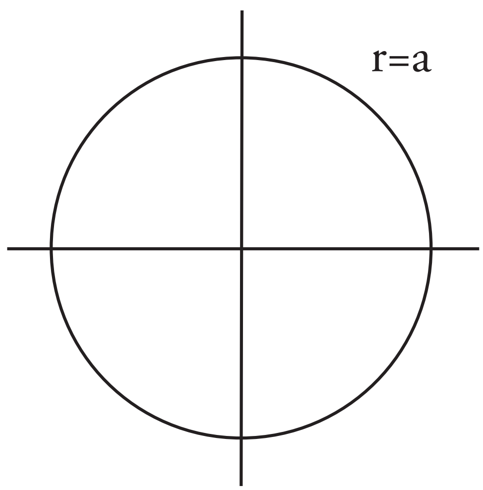

Figure 8: Example 4: Circle centered at the origin

$$x = r\cos\theta;\ y = r\sin\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

The circle is $x^2 + y^2 = a^2$, so $r = a$ and

$$x = a\cos\theta;\ y = a\sin\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}a^2\, d\theta = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2.$$

Example 5: A Ray. In this case, $\theta = b$.

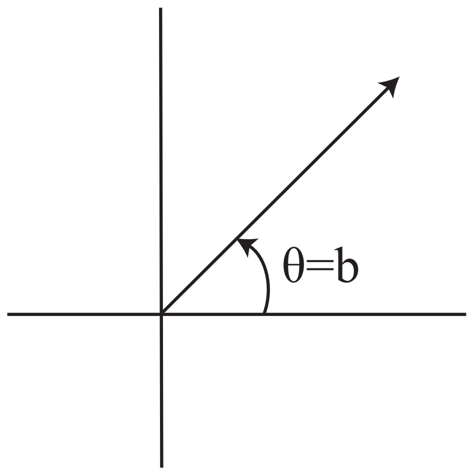

Figure 9: Example 5: The ray $\theta = b, 0 \leq r < \infty$.

The range of r is $0 \leq r < \infty$; $x = r\cos b$; $y = r\sin b$.

Example 6: Finding the Polar Formula, based on the Cartesian Formula

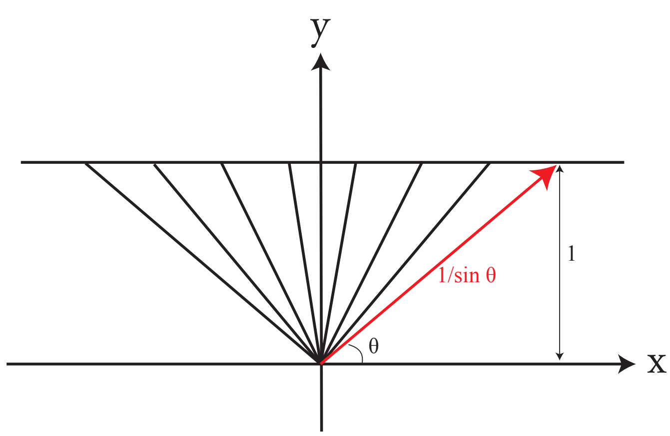

Figure 10: Example 6: Cartesian Form to Polar Form

Consider, in cartesian coordinates, the line $y = 1$. To find the polar coordinate equation, plug in $y = r\sin\theta$ and $x = r\cos\theta$ and solve for r.

$$r\sin\theta = 1 \implies r = \frac{1}{\sin\theta} \quad \text{with} \quad 0 < \theta < \pi$$

Example 7: Going back to (x, y) coordinates from $r = f(\theta)$.
Start with

$$r = \frac{1}{1 + \frac{1}{2}\sin\theta}.$$

Hence,

$$r + \frac{r}{2}\sin\theta = 1$$

Plug in $r = \sqrt{x^2 + y^2}$:

$$\sqrt{x^2 + y^2} + \frac{y}{2} = 1$$

$$\sqrt{x^2 + y^2} = 1 - \frac{y}{2} \qquad \Longrightarrow \qquad x^2 + y^2 = \left(1 - \frac{y}{2}\right)^2 = 1 - y + \frac{y^2}{4}$$

Finally,

$$x^2 + \frac{3y^2}{4} + y = 1$$

This is an equation for an ellipse, with the origin at one focus.

Useful conversion formulas:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example 8: A Rose $r = \cos(2\theta)$
The graph looks a bit like a flower:

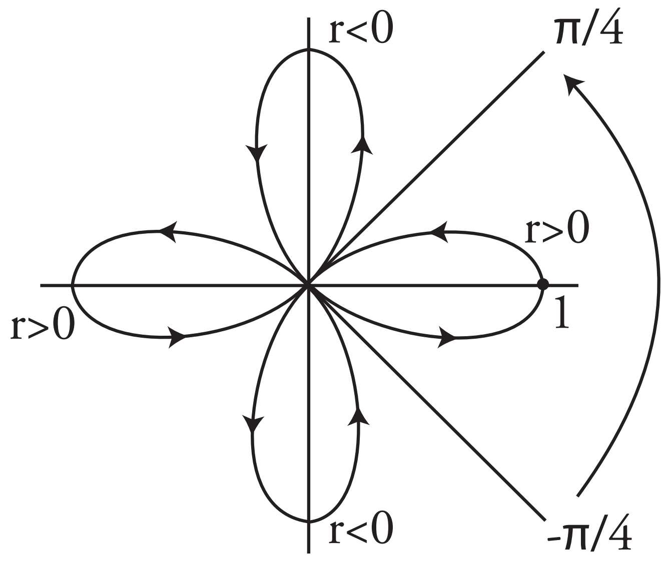

Figure 11: Example 8: Rose

For the first "petal"

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

Note: Next lecture is Lecture 34 as Lecture 33 is Exam 4.