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18.01 Single Variable Calculus  
Fall 2006

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## Exam 4 Review

1. Trig substitution and trig integrals.
2. Partial fractions.
3. Integration by parts.
4. Arc length and surface area of revolution
5. Polar coordinates
6. Area in polar coordinates.

### Questions from the Students

- Q: What do we need to know about parametric equations?
- A: Just keep this formula in mind:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example: You're given

$$x(t) = t^4$$

and

$$y(t) = 1 + t$$

Find  $s$  (length).

$$ds = \sqrt{(4t^3)^2 + (1)^2} dt$$

Then, integrate with respect to  $t$ .

- Q: Can you quickly review how to do partial fractions?
- A: When finding partial fractions, first check whether the degree of the numerator is greater than or equal to the degree of the denominator. If so, you first need to do algebraic long-division. If not, then you can split into partial fractions.

**Example.**

$$\frac{x^2 + x + 1}{(x - 1)^2(x + 2)}$$

We already know the *form* of the solution:

$$\frac{x^2 + x + 1}{(x - 1)^2(x + 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 2}$$

There are two coefficients that are easy to find:  $B$  and  $C$ . We can find these by the cover-up method.

$$B = \frac{1^2 + 1 + 1}{1 + 2} = \frac{3}{3} \quad (x \rightarrow 1)$$

To find  $C$ ,

$$C = \frac{(-2)^2 - 2 + 1}{(-2 - 1)^2} = \frac{1}{3} \quad (x \rightarrow -2)$$

To find  $A$ , one method is to plug in the easiest value of  $x$  other than the ones we already used ( $x = 1, -2$ ). Usually, we use  $x = 0$ .

$$\frac{1}{(-1)^2(2)} = \frac{A}{-1} + \frac{1}{(-1)^2} + \frac{1/3}{2}$$

and then solve to find  $A$ .

The Review Sheet handed out during lecture follows on the next page.

## Exam 4 Review Handout

1. Integrate by **trigonometric substitution**; evaluate the **trigonometric integral** and work backwards to the original variable by evaluating **trig(trig<sup>-1</sup>)** using a right triangle:

- a)  $a^2 - x^2$  use  $x = a \sin u$ ,  $dx = a \cos u \, du$ .
- b)  $a^2 + x^2$  use  $x = a \tan u$ ,  $dx = a \sec^2 u \, du$
- c)  $x^2 - a^2$  use  $x = a \sec u$ ,  $dx = a \sec u \tan u \, du$

2. Integrate rational functions  $P/Q$  (ratio of polynomials) by the method of **partial fractions**: If the degree of  $P$  is less than the degree of  $Q$ , then factor  $Q$  completely into linear and quadratic factors, and write  $P/Q$  as a sum of simpler terms. For example,

$$\frac{3x^2 + 1}{(x-1)(x+2)^2(x^2+9)} = \frac{A}{x-1} + \frac{B_1}{x+2} + \frac{B_2}{(x+2)^2} + \frac{Cx+D}{x^2+9}$$

Terms such as  $D/(x^2+9)$  can be integrated using the trigonometric substitution  $x = 3 \tan u$ .

This method can be used to evaluate the integral of any rational function. In practice, the hard part turns out to be factoring the denominator! In recitation you encountered two other steps required to cover every case systematically, namely, completing the square<sup>1</sup> and long division.<sup>2</sup>

### 3. Integration by parts:

$$\int_a^b uv' \, dx = uv \Big|_a^b - \int_a^b u'v \, dx$$

This is used when  $u'v$  is simpler than  $uv'$ . (This is often the case if  $u'$  is simpler than  $u$ .)

4. **Arclength**:  $ds = \sqrt{dx^2 + dy^2}$ . Depending on whether you want to integrate with respect to  $x$ ,  $t$  or  $y$  this is written

$$ds = \sqrt{1 + (dy/dx)^2} \, dx; \quad ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt; \quad ds = \sqrt{(dx/dy)^2 + 1} \, dy$$

### 5. Surface area for a surface of revolution:

- a) around the  $x$ -axis:  $2\pi y \, ds = 2\pi y \sqrt{1 + (dy/dx)^2} \, dx$  (requires a formula for  $y = y(x)$ )
- b) around the  $y$ -axis:  $2\pi x \, ds = 2\pi x \sqrt{(dx/dy)^2 + 1} \, dy$  (requires a formula for  $x = x(y)$ )

### 6. Polar coordinates: $x = r \cos \theta$ , $y = r \sin \theta$ (or, more rarely, $r = \sqrt{x^2 + y^2}$ , $\theta = \tan^{-1}(y/x)$ )

- a) Find the polar equation for a curve from its equation in  $(x, y)$  variables by substitution.
- b) Sketch curves given in polar coordinates and understand the range of the variable  $\theta$  (often in preparation for integration).

### 7. Area in polar coordinates:

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

(Pay attention to the range of  $\theta$  to be sure that you are not double-counting regions or missing them.)

<sup>1</sup>For example, we rewrite the denominator  $x^2 + 4x + 13 = (x+2)^2 + 9 = u^2 + a^2$  with  $u = x+2$  and  $a = 3$ .

<sup>2</sup>Long division is used when the degree of  $P$  is greater than or equal to the degree of  $Q$ . It expresses  $P(x)/Q(x) = P_1(x) + R(x)/Q(x)$  with  $P_1$  a quotient polynomial (easy to integrate) and  $R$  a remainder. The key point is that the remainder  $R$  has degree less than  $Q$ , so  $R/Q$  can be split into partial fractions.

The following formulas will be printed with Exam 4

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1; & \sec^2 x &= \tan^2 x + 1 \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x; & \cos^2 x &= \frac{1}{2} + \frac{1}{2} \cos 2x \\ \cos 2x &= \cos^2 x - \sin^2 x; & \sin 2x &= 2 \sin x \cos x\end{aligned}$$

$$\frac{d}{dx} \tan x = \sec^2 x; \quad \frac{d}{dx} \sec x = \sec x \tan x; \quad \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}; \quad \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \tan x \, dx = -\ln(\cos x) + c; \quad \int \sec x \, dx = \ln(\sec x + \tan x) + c$$

See the next page for a review on integration of rational functions.

## Postscript: Systematic integration of rational functions

For a general rational function  $P/Q$ , the first step is to express  $P/Q$  as the sum of a polynomial and a ratio in which the numerator has smaller degree than the denominator.

For example,

$$\frac{x^3}{x^2 - 2x + 1} = x + 2 + \frac{3x - 2}{x^2 - 2x + 1}$$

(To carry out this long division, do not factor the denominator  $Q(x) = x^2 - 2x + 1$ , just leave it alone.) The quotient  $x + 2$  is a polynomial and is easy to integrate. The remainder term

$$\frac{3x - 2}{(x - 1)^2}$$

has a numerator  $3x - 2$  of degree 1 which is less than the degree 2 of the denominator  $(x - 1)^2$ . Therefore there is a partial fraction decomposition. In fact,

$$\frac{3x - 2}{(x - 1)^2} = \frac{(3x - 3) + 1}{(x - 1)^2} = \frac{3}{x - 1} + \frac{1}{(x - 1)^2}$$

In general, if  $P$  has degree  $n$  and  $Q$  has degree  $m$ , then long division gives

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)}$$

in which  $P_1$ , the quotient in the long division, has degree  $n - m$  and  $R$ , the remainder in the long division, has degree at most  $m - 1$ .

### Evaluation of the “simple” pieces

The integral

$$\int \frac{dx}{(x - a)^n} = \frac{-1}{n - 1} (x - a)^{1-n} + c$$

if  $n \neq 1$  and  $\ln|x - a| + c$  if  $n = 1$ . On the other hand the terms

$$\int \frac{xdx}{(Ax^2 + Bx + C)^n} \quad \text{and} \quad \int \frac{dx}{(Ax^2 + Bx + C)^n}$$

are handled by first completing the square:

$$Ax^2 + Bx + C = A(x - B/2A)^2 + \left(C - \frac{B^2}{4A}\right)$$

Using the variable  $u = \sqrt{A}(x - B/2A)$  yields combinations of integrals of the form

$$\int \frac{udu}{(u^2 + k^2)^n} \quad \text{and} \quad \int \frac{du}{(u^2 + k^2)^n}$$

The first integral is handled by the substitution  $w = u^2 + k^2$ ,  $dw = 2udu$ . The second integral can be worked out using the trigonometric substitution  $u = k \tan \theta$   $du = k \sec^2 \theta d\theta$ . This then leads to sec-tan integrals, and the actual computation for large values of  $n$  are long.

There are also other cases that we will not cover systematically. Examples are below:

1. If  $Q(x) = (x - a)^m(x - b)^n$ , then the expression is

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_m}{(x - a)^m} + \frac{B_1}{x - b} + \frac{B_2}{(x - b)^2} + \cdots + \frac{B_n}{(x - b)^n}$$

2. If there are quadratic factors like  $(Ax^2 + Bx + C)^p$ , one gets terms

$$\frac{a_1x + b_1}{Ax^2 + Bx + C} + \frac{a_2x + b_2x}{(Ax^2 + Bx + C)^2} + \cdots + \frac{a_px + b_p}{(Ax^2 + Bx + C)^p}$$

for each such factor. (To integrate these quadratic pieces complete the square and make a trigonometric substitution.)