Lecture 31. December 8, 2005

Practice Problems. Course Reader: 7B-4, 7B-6, 7C-1, 7C-5, 7D-1, 7D-2.

1. Power series. Given a real number a and a sequence of real numbers $(c_n)_{n\geq 0}$, there is an associated expression, called a *power series about $x = a$*,

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

For every choice of a real number x, the power series gives a usual series. In particular, for the choice $x = a$, the series has only 1 nonzero term, thus converges to c_0.

Question. Given a power series, for which real numbers x does the corresponding series absolutely converge?

Examples. 1. Consider the power series,

$$0 + 1^1 x^1 + 2^2 x^2 + 3^3 x^3 + \cdots = \sum_{n=1}^{\infty} n^n x^n.$$

Of course this converges to 0 for $x = 0$. But for any x other than 0, the sequence $n^n x^n = (nx)^n$ diverges. Therefore the series does not converge. In other words, the series converges only for $x = 0$.

2. Consider the power series,

$$1 + x + x^2 + \cdots = \sum_{n=1}^{\infty} x^n.$$

This is a geometric series. From the last lecture, the series converges absolutely for $|x| < 1$ and diverges if $|x| \geq 1$.

3. Consider the power series,

$$1 + x + x^2/2 + x^3/3! + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

The ratio of the nth and $(n+1)^{\text{st}}$ terms in the series is,

$$(x^{n+1}/(n+1)!)/(x^n/n!) = \frac{x}{n+1}.$$

For fixed x, as n grows, this sequence of ratios converges to 0, which is less than 1. Therefore, by the ratio test, for every choice of x the series converges.

These 3 examples illustrate the whole range of possibilities.

Theorem. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series about $x = a$. Exactly one of the following hold.

(i) For every x different from a, the series does not converge absolutely.

(ii) There exists a real number R such that the series converges absolutely if $|x - a| < R$ and does not converge absolutely if $|x - a| > R$.

(iii) For every real number x, the series converges absolutely.

The real number R occuring in Case (ii) is called the *radius of convergence*. By convention, in Case (i) the radius of convergence is defined to be $R = 0$. By convention, in Case (iii) the radius of convergence is defined to be $R = \infty$. This allows us to replace the original question by a more precise question.

Question. Given a power series, what is the radius of convergence?

Although there is no single answer to this question, in many interesting cases the ratio or root test gives an answer.

2. Analytic functions. If the radius of convergence R of a power series $\sum c_n(x-a)^n$ is positive, then the power series defines a function on the interval $(a - R, a + R)$,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

A function defined in this manner is called an *analytic function*. This is the real significance of power series: they give important examples of functions that cannot be described in a more direct manner. Analytic functions have nice analytic properties (whence the name). For instance, it is a theorem (proved in 18.100) that an analytic function $f(x)$ is differentiable and the derivative has a power series converging absolutely with the same radius R,

$$f'(x) = \sum_{n=0}^{\infty} c_n n(x-a)^{n-1} = \sum_{m=0}^{\infty} (m+1)c_{m+1}(x-a)^m.$$

We can iterate the theorem, i.e., $f'(x)$ is differentiable and $f''(x)$ has a power series converging absolutely with radius R. Iterating k times, the function $f(x)$ is k-times differentiable and its kth derivative has a power series,

$$f^{(k)}(x) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} c_{n+k}(x-a)^n.$$

In particular, every derivative of $f(x)$ is defined. A function with this property is called *infinitely differentiable* or *smooth*. Thus, every analytic function is infinitely differentiable.

This is only 1 of many useful properties of analytic functions. Which functions $f(x)$ are analytic functions? By the last paragraph, if $f(x)$ is analytic, then it is infinitely differentiable. Are there other restrictions? Can more than 1 power series about $x = a$ give rise to the same analytic function?

To answer both of these questions, consider the analytic function defined by a power series,

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n.$$

Plugging in $x = a$ gives the equation,

$$f(a) = c_0 + c_1(a - a) + c_2(a - a)^2 + \cdots = c_0 + 0 + 0 + \cdots = c_0.$$

Thus the first coefficient of the power series is simply,

$$c_0 = f(a).$$

Moreover, from the power series for the kth derivative,

$$f^{(k)}(a) = k!c_k + (k+1)!/1!c_{k+1}(a - a) + (k+2)!/2!c_{k+2}(a - a)^2 + \cdots = k!c_k + 0 + 0 + \cdots = k!c_k.$$

Solving for c_k, the kth coefficient of the power series is,

$$c_k = \boxed{f^{(k)}(a)/k!.}$$

Therefore, the power series defining $f(x)$ is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

In particular, this series is unique. This answers the second question. Two absolutely convergent power series about $x = a$ give the same analytic function if and only if the power series are themselves equal (i.e., the corresponding coefficients of the 2 series are equal).

Moreover, this gives us alot of information about the first question. For an infinitely differentiable function $f(x)$ defined at a point $x = a$, there is a very important power series, the *Taylor series expansion of $f(x)$ about $x = a$*,

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.}$$

If $f(x)$ is analytic, then the Taylor series converges absolutely to $f(x)$. This reduces the original question to 2 new questions. Does the Taylor series have a positive radius of convergence? If so, does the analytic function defined in this way equal the original function $f(x)$?

The radius of convergence question is precisely the radius of convergence question posed earlier. As there, the answer can often be found by using the ratio or root tests. The second question is yes in every practical case. There are examples of infinitely differentiable functions where the Taylor series has a positive radius of convergence, but does not converge to the original function. However, every example is somewhat contrived; they rarely come up "in nature". Just for completeness, here is an example of one of these pathological functions,

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0 & x = 0 \end{cases}$$

3. Algorithm for computing Taylor series. The method for finding the Taylor series of a function is always the same. For definiteness, consider the Taylor series expansion of $f(x) = (1-x)^{-1}$ about the point $x = 0$.

Step 1. Compute all derivatives of $f(x)$. If this sounds like alot of work, it is! In most examples, this really comes down to finding an inductive formula for the derivatives of $f(x)$. In the example, the "zeroth derivative" is,

$$f(x) = \boxed{(1-x)^{-1}.}$$

The first derivative is,

$$f'(x) = -(1-x)^{-2}(-1) = \boxed{(1-x)^{-2}.}$$

The second derivative is,

$$f''(x) = (-2)(1-x)^{-3}(-1) = \boxed{(1-x)^{-3}.}$$

This begins to suggest a pattern: The kth derivative of $f(x)$ will be,

$$f^{(k)}(x) = b_k(1-x)^{-k-1},$$

for some real number b_k. Having made this guess, it is easy to verify by induction. By computation, the result is true for $k = 0, 1$ and 2 with the corresponding real numbers $b_0 = 1$, $b_1 = 1$ and $b_2 = 2$. By way of induction, assume the result is true for $k = n$, i.e.,

$$f^{(n)}(x) = c_n(1-x)^{-n-1}.$$

Then the $(n+1)^{\text{st}}$ derivative is,

$$f^{(n+1)}(x) = (f^{(n)}(x))' = (c_n(1-x)^{-n-1})' = c_n(-n-1)(1-x)^{-n-2}(-1) = (n+1)c_n(1-x)^{-n-2}.$$

Thus the result is also true for $k = n+1$ where c_{n+1} satisfies the equation,

$$c_{n+1} = (n+1)c_n.$$

Thus the result is proved by induction on k.

In fact, more has been accomplished, since now there is an inductive formula for the numbers c_n,

$$c_n = nc_{n-1} = n(n-1)c_{n-2} = n(n-1)(n-2)c_{n-3} = \cdots = n(n-1)(n-2)\cdots\cdots3c_2 = n(n-1)(n-2)\cdots\cdots3\cdot2\cdot1.$$

This number has come up before in this class. It is the nth factorial number,

$$c_n = n!.$$

This gives the final formula for the nth derivative of $f(x)$,

$$f^{(n)}(x) = \boxed{n!(1-x)^{-n-1}.}$$

Step 2. Substitute $x = a$ into the derivatives. Compared to the work of finding the derivatives, this is very simple. In the example, plugging in $x = 0$ gives,

$$f^{(n)}(0) = \boxed{n!.}$$

Step 3. Compute the coefficients of the Taylor series. By definition, the nth coefficient of the Taylor series is,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

In the example, this gives the coefficient,

$$c_n = \frac{n!}{n!} = \boxed{1,}$$

for every integer $n \geq 0$.

Step 4. Write the Taylor series. This is really getting into the "mind-numbing details". In the example, the Taylor series expansion for $(1-x)^{-1}$ about $x = 0$ is,

$$(1-x)^{-1} = \boxed{\sum_{n=0}^{\infty} x^n.}$$

Step 5. If possible, find the radius of convergence. In the example, the Taylor series is simply the geometric series. By the previous lecture, the geometric series converges absolutely with radius $R = 1$. Moreover, it converges absolutely to $(1-x)^{-1}$. Notice, this gives another explanation for the radius $R = 1$. Since $(1-x)^{-1}$ has a vertical asymptote at $x = 1$, the Taylor series cannot converge on any interval that contains $x = 1$. The largest interval centered at $x = 0$ not containing $x = 1$ is the interval $(-1, 1)$. This interval has radius $R = 1$.

4. More examples. What is the Taylor series expansion for $(1-x)^{-1}$ about a point $x = a$ different from $x = 1$? The fortunate fact is that Step 1 allows to compute the derivatives $f^{(n)}(a)$

for any $a \neq 1$, not just $x = 0$. This is the typical case, and it is one justification for doing the work necessary in Step 1. In this case, the answer is,

$$f^{(n)}(a) = \boxed{n!(1-a)^{-n-1}.}$$

Therefore, according the Step 3, the nth coefficient in the Taylor series expansion is,

$$c_n = \frac{n!(1-a)^{-n-1}}{n!} = \boxed{(1-a)^{-n-1}.}$$

Thus, according to Step 4, the Taylor series expansion for $(1-x)^{-1}$ about $x = a$ is,

$$(1-x)^{-1} = \boxed{\sum_{n=0}^{\infty} (1-a)^{-n-1}(x-a)^n.}$$

What is the radius of convergence? The ratio of the $(n+1)^{\text{st}}$ and n^{th} terms of the series is,

$$[(1-a)^{-n-2}(x-a)^{n+1}]/[(1-a)^{-n-1}(x-a)^n] = (1-a)^{-1}(x-a).$$

This is independent of n. Thus, this constant sequence converges to its constant value $(1-a)^{-1}(x-a)$. By the ratio test, the sequence is absolutely convergent if and only if this limit has absolute value less than 1,

$$|(1-a)^{-1}(x-a)| \leq 1.$$

Rearranging, the series converges if and only if,

$$|x-a| \leq |1-a|.$$

Thus the radius of convergence is,

$$R = \boxed{|1-a|.}$$

This is perfectly reasonable. The function $(1-x)^{-1}$ has a vertical asymptote at $x = 1$. Therefore, the power series cannot converge on any interval containing $x = 1$. The largest interval centered at $x = a$ not containing $x = 1$ has radius equal to the distance from $x = a$ to $x = 1$, namely $R = |1-a|$.

Example 2. For the next example, consider the Taylor series expansion for $f(x) = e^x$ near $x = 0$. In this case, Step 1 is simple. Every derivative of $f(x)$ is simply,

$$f^{(n)}(x) = \boxed{e^x.}$$

Therefore, the nth coefficient of the Taylor series expansion is,

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \boxed{1/n!.}$$

Therefore the Taylor series expansion is,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Observe this is the power series considered earlier in the lecture, whose radius of convergence is $R = \infty$. Therefore, for every x, the power series converges absolutely to e^x,

$$e^x = \boxed{\sum_{n=0}^{\infty} x^n/n!.}$$

This equation is sometimes taken as the definition of e^x. It has certain advantages to our original definition of e^x. Importantly, it is easy for a computer to determine e^x to very high precision using this formula.

Example 3. Having computed the Taylor series expansion for e^x about $x = 0$, the next question is to compute the Taylor series expansion for e^x about $x = a$. According to the formula,

$$f^{(n)}(a) = e^a,$$

and thus the coefficient is,

$$c + n = \boxed{e^a/n!.}$$

This gives the Taylor series expansion for e^x about $x = a$,

$$\sum_{n=0}^{\infty} \frac{e^a}{n!} (x - a)^n.$$

As above, the radius of convergence is $R = \infty$. Thus, for every real number x, the power series converges absolutely to e^x,

$$e^x = \boxed{\sum_{n=0}^{\infty} e^a (x - a)^n/n!.}$$

On the other hand, we didn't need to do any extra work to see this. We could have used the formula,

$$e^x = e^{a+(x-a)} = e^a e^{x-a}.$$

Plugging in $x - a$ for x in the power series expansion for e^x gives the power series expansion,

$$e^{x-a} = \sum_{n=0}^{\infty} \frac{1}{n!} (x - a)^n.$$

This gives the same Taylor series expansion as above,

$$e^x = e^a \sum_{n=0}^{\infty} \frac{1}{n!} (x - a)^n = \boxed{\sum_{n=0}^{\infty} (e^a/n!)(x - a)^n.}$$

Example 3. Consider the function $f(x) = \sin(x)$. The derivatives of $f(x)$ are,

$$
\begin{array}{rcl}
f(x) & = & \sin(x), \\
f'(x) & = & \cos(x), \\
f''(x) & = & -\sin(x), \\
f^{(3)}(x) & = & -\cos(x), \\
f^{(n+4)}(x) & = & f^{(n)}(x)
\end{array}
$$

Together, these give all the derivatives of $f(x)$. Write $n = 4l$, $4l + 1$, $4l + 2$ or $4l + 3$ for some nonnegative integer l. Then the rules above give,

$$
f^{(n)}(x) = \begin{cases}
\sin(x) & n = 4l, \\
\cos(x) & n = 4l + 1, \\
-\sin(x) & n = 4l + 2, \\
-\cos(x) & n = 4l + 3
\end{cases}
$$

In particular, plugging in $x = 0$ gives,

$$
f^{(n)}(0) = \begin{cases}
0, & n = 4l, \\
1, & n = 4l + 1, \\
0, & n = 4l + 2, \\
-1, & n = 4l + 3
\end{cases}
$$

Thus, all the even coefficients of the Taylor series are 0. For an odd coefficient, say $n = 2m + 1$, the derivative is,

$$
f^{(2m+1)}(0) = (-1)^m.
$$

Therefore, the coefficient is,

$$
c_{2m+1} = \frac{(-1)^m}{(2m + 1)!}.
$$

Plugging this in gives the Taylor series expansion for $\sin(x)$ about $x = 0$,

$$
\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)!} x^{2m+1}.
$$

The ratio of consecutive terms in the series is,

$$
[(-1)^{m+1} x^{2m+3}/(2m + 3)!]/[(-1)^m x^{2m+1}/(2m + 1)!] = -x^2/(4m^2 + 8m + 3).
$$

This sequence converges to 0. Therefore, by the ratio test, the power series converges absolutely to $\sin(x)$ for every choice of x,

$$
\sin(x) = \boxed{\sum_{m=0}^{\infty} (-1)^m/(2m + 1)! x^{2m+1}.}
$$

There is an exactly similar formula for $g(x) = \cos(x)$,

$$g^{(n)}(x) = \begin{cases} \cos(x), & n = 4l, \\ -\sin(x), & n = 4l + 1, \\ -\cos(x), & n = 4l + 2, \\ \sin(x), & n = 4l + 3. \end{cases}$$

This gives the values,

$$g^{(n)}(0) = \begin{cases} 1, & n = 4l, \\ 0, & n = 4l + 1, \\ -1, & n = 4l + 2, \\ 0, & n = 4l + 3. \end{cases}$$

Therefore the Taylor series is,

$$\cos(x) = \boxed{\sum_{m=0}^{\infty} (-1)^m / (2m)! x^{2m}.}$$

Notice, we didn't really need to do this work. Since $\cos(x)$ is the derivative of $\sin(x)$, the Taylor series for $\cos(x)$ is simply the term-by-term derivative of the Taylor series for $\sin(x)$. This gives the same formula as above.

To compute the Taylor series expansions of $\sin(x)$ and $\cos(x)$ about a point $x = a$, we can follow the procedure above. However, it is much faster to use the angle addition formulas,

$$\sin(x) = \sin(a + (x - a)) = \cos(a)\sin(x - a) + \sin(a)\cos(x - a),$$

$$\cos(x) = \cos(a + (x - a)) = \cos(a)\cos(x - a) - \sin(a)\sin(x - a).$$

This gives the Taylor series expansions,

$$\sin(x) = \boxed{\sum_{m=0}^{\infty} \frac{(-1)^m \cos(a)}{(2m+1)!}(x-a)^{2m+1} + \sum_{m=0}^{\infty} \frac{(-1)^m \sin(a)}{(2m)!}(x-a)^{2m},}$$

$$\cos(x) = \boxed{\sum_{m=0}^{\infty} \frac{(-1)^m \cos(a)}{(2m)!}(x-a)^{2m} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \sin(a)}{(2m+1)!}(x-a)^{2m+1}.}$$