Solutions to Problem Set 5

Late homework policy. Late work will be accepted only with a medical note or for another Institute-approved reason.

Cooperation policy. You are encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding. You may not copy another student's solutions. And you should not refer to notes from a study group while writing up your solutions (if you need to refer to notes from a study group, it isn't really "your own understanding").

Part I. These problems are mostly from the textbook and reinforce the basic techniques. Occasionally the solution to a problem will be in the back of the textbook. In that case, you should work the problem first and only use the solution to check your answer.

Part II. These problems are not taken from the textbook. They are more difficult and are worth more points. When you are asked to "show" some fact, you are not expected to write a "rigorous solution" in the mathematician's sense, nor a "textbook solution". However, you should write a clear argument, using English words and complete sentences, that would convince a typical Calculus student. (Run your argument by a classmate; this is a good way to see if your argument is reasonable.) Also, for the grader's sake, try to keep your answers as short as possible (but don't leave out *important* steps).

Part I(20 points)
    (a) (4 points) p. 181, Section 5.4,   Problem 12
    (b) (4 points) p. 283, Section 8.5,   Problem 16
    (c) (4 points) p. 374, Section 10.9, Problem 4
    (d) (4 points) p. 225, Section 7.2,   Problem 25
    (e) (4 points) p. 229, Section 7.3,   Problem 7

Solution to (a) We rewrite the equation $\frac{dy}{dx} = \frac{5+3x^2}{2+2y}$ in form

$$(2 + 2y)dy = (5 + 3x^2)dx$$

and integrate both sides to obtain $\int(2 + 2y)dy = \int(5 + 3x^2)dx$, or

$$y^2 + 2y = x^3 + 5x + C$$

where C is a constant, which is going to be determined by the initial condition $y = 3$ when $x = 2$. Thus we have $3^2 + 2.3 = 2^3 + 5.2 + C$ which yields $C = 26$. In order to obtain an expression for y,

we complete the left hand side to a complete square,

$$y^2 + 2y + 1 - 1 = (y+1)^2 - 1 = x^3 + 5x + C = x^3 + 5x + 26$$

or $(y+1)^2 = x^3 + 5x + 27$, therefore

$$\boxed{y = \sqrt{x^3 + 5x + 27} - 1}$$

Note that we chose the plus sign for the square root to be consistent with the given initial condition.

Solution to (b) We are given the equation $L\frac{dI}{dt} = E - RI$, which is separated as

$$\frac{LdI}{E - RI} = dt$$

In order to integrate this expression, we make the substitution $u = E - RI$, $du = -RdI$, where $I = \frac{E}{R} - \frac{1}{R}u$, which transforms the integral in question to

$$-\frac{L}{R}\frac{du}{u} = dt$$

which integrates to

$$-\frac{L}{R}\ln|u| = t + C \Rightarrow u = u_0 e^{-\frac{L}{R}t}$$

So for a general initial current $I(0) = I_0$, the current is given by

$$I(t) = \frac{E}{R} + (I_0 - \frac{E}{R})e^{-\frac{R}{L}t}.$$

In particular, the initial condition $I_0 = 0$ gives the solution,

$$I(t) = \boxed{(E/R)(1 - e^{-Rt/L}).}$$

The graph of this current, as well as for several other choices of I_0, are on the course webpage.

Solution to (c) Conveniently, $n = 4$ corresponds to evaluating the function at the points $1, 2, 3, 4, 5$:

$$
\begin{array}{ll}
x_0 = 1 & y_0 = e \\
x_1 = 2 & y_1 = \frac{1}{2}e^2 \\
x_2 = 3 & y_2 = \frac{1}{3} \\
x_3 = 4 & y_3 = \frac{1}{4}e^4 \\
x_4 = 5 & y_4 = \frac{1}{5}e^5
\end{array}
$$

Note that $b - a = 5 - 1 = 4$, and $4 = n = 2m \rightarrow m = 2$. The Simpson formula then given

$$I_{simpson} = \frac{b-a}{6m}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = \frac{4}{12}(e + 2e^2 + \frac{2}{3}e^3 + e^4 + \frac{e^5}{5}) = \boxed{38.4}$$

Solution to (d) We first need to calculate the tangent line of $y = x^3$ at $x = 1$. As the slope of this line is given by $y'(1) = 3x^2|_{x=1} = 3$, it is given by

$$y = 3(x - 1) + 1 = 3x - 2 \tag{1}$$

A graph of the region is included on the webpage.

In order to calculate the area between this tangent line and the graph of $y = f(x)$ we need to find the other point where the two intersect. Let's denote this point by (u, u^3) (remember that this point is on $y = x^3$) . Then, by plugging into (1), we obtain $u^3 = 3u - 2$. This can be rewritten as

$$u^3 - 3u + 2 = 0 \tag{2}$$

which, at first sight, not easy to solve. However, since the tangent line touches the curve at $u = 1$, we know one solution is $u = 1$. In fact, $u = 1$ is a double root, because the tangent line is tangent to the curve at that point. Therefore, we should be able to rewrite (2) in the form

$$u^3 - 3u + 2 = (u - 1)^2(u - a)$$

where a is the other root. By multiplying out, or a simple inspection, we easily see that $a = -2$. Therefore, the sought integral has the limits -2 and 1. Hence

$$A = \int_{-2}^{1} [x^3 - (3x - 2)]dx = (\frac{x^4}{4} - \frac{3x^2}{2} + 2x)|_{-2}^{1} = \boxed{\frac{27}{4}}$$

Solution to (e) From the figure on the course webpage, each section has volume $dV = Adx = h^2 dx$, and we have the constraint $x^2 + h^2 = a^2$. Therefore the total volume is given by

$$V = \int_{-a}^{a} h^2 dx = \int_{-a}^{a} (a^2 - x^2)dx = a^2 x + \frac{x^3}{3}|_{-a}^{a} = \boxed{\frac{4}{3}a^3}$$

Part II(30 points)

Problem 1(15 points) For each positive integer n, consider the collection of curves,

$$y = Cx^n,$$

where C varies over all real numbers.

(a)(5 points) For $y = Cx^n$, find the derivative dy/dx. Then use the expression $y = Cx^n$ to eliminate the constant C from dy/dx. Write your answer in the form,

$$\frac{dy}{dx} = F(x, y),$$

for some function F that does not involve C.

Solution to (a) By differentiating $y = Cx^n$, we obtain $\frac{dy}{dx} = Cnx^{n-1}$. One way to get rid of C is to write

$$C = \frac{y}{x^n} = \frac{dy}{dx}\frac{1}{nx^{n-1}}$$

where we used both our equations to obtain expressions for C. Thus, we have obtained

$$\frac{dy}{dx} = \frac{ny}{x}$$

which is the required differential equation. Note that this equation does not involve C, and it is satisfied by $y = Cx^n$ for any arbitrary C.

(b)(5 points) A differentiable function $y = g(x)$ has the property that for every point (a, b) on the graph, the tangent line to $y = g(x)$ at (a, b) is orthogonal to the tangent line to $y = Cx^n$ at (a, b). Here C is chosen to be,

$$C = \frac{b}{a^n},$$

so that (a, b) is on the graph of $y = Cx^n$. Use the hypothesis to write a differential equation $g(x)$ satisfies.

Hint. The differential equation is *separable.*

Solution to (b) We remember the slopes of the two orthogonal lines multiply to -1. It's easy to read the slope of the tangent line to be $\frac{ny}{x}$. Therefore the sought equation is

$$\boxed{\frac{dy}{dx} = -\frac{1}{\frac{ny}{x}} = -\frac{x}{ny}} \tag{3}$$

(c)(5 points) Solve the separable differential equation to find all curves $y = g(x)$ satisfying the hypothesis. Express the curve implicitly in the form,

$$G(x, y) = E,$$

where $G(x, y)$ is a polynomial, and E is a positive real number.

Solution to (c) The equation (3) can be separated as $nydy = -xdx$, therefore

$$\int nydy = -\int xdx$$

or $\frac{n}{2}y^2 = -\frac{1}{2} + C$ where C is an arbitrary constant. We can rewrite this as

$$\boxed{x^2 + ny^2 = E}$$

where $E = 2C$ is another arbitrary constant.

Problem 2(10 points) Solve Problem 11 on p. 229 of the textbook. The answer is in the back of the book (and even in the statement of the problem). You are responsible for showing the work, and explaining the steps, leading to the answer.

Solution to Problem 2 We are considering the volume generated by revolving the given circle about y-axis. An illustration of the torus, with a washer highlighted, is available on the course webpage. By the washer method, $dV = A dy = \pi(r_o^2 - r_i^2)$. The given constraint is $(x-b)^2 + y^2 = a^2$, therefore $x = b \pm \sqrt{a^2 - y^2}$. We infer that

$$r_o = b + \sqrt{a^2 - y^2} \qquad r_i = b - \sqrt{a^2 - y^2}.$$

Therefore $r_0^2 - r_i^2 = (r_i + r_o)(r_i - r_i) = 2b.2\sqrt{a^2 - y^2}$. Hence

$$V = \int_{y=-a}^{y=b} \pi(r_o^2 - r_i^2) = \pi \int_{y=-a}^{y=b} 4b\sqrt{a^2 - y^2}dy = 4\pi b \int_{y=-a}^{y=b} \sqrt{a^2 - y^2}dy = 4\pi b (\frac{1}{2}\pi a^2) = \boxed{2\pi^2 b a^2}$$

where we used the fact that the integral $\int_{-a}^{a} \sqrt{a^2 - y^2}dy$ is equal to the area of the upper half of the circle of radius a, which is equal to $\frac{1}{2}\pi a^2$.

Problem 3(5 points) A *cone* over a plane region R is the solid obtained by joining every point in R to a fixed point P not in the plane. If the height of P above the plane is h, and if the area of the region R is A, the volume of the cone is,

$$V = \frac{1}{3}Ah.$$

Prove this fact by computing the volume using the washer method. You may use the fact that if a plane region is scaled by a factor c, the area scales by the factor c^2.

Solution to Problem 3 An illustration of the setup, with a slice highlighted, is available on the course webpage. If we declare area of the section at height z to be $A(z)$, then the volume would be given by

$$V = \int_0^h A(z)dx$$

We can use the fact given in the hint, which reads in mathematical symbols as

$$A(z) = s^2 A$$

where s is a scale factor which decreases linearly from 1 at $z = 0$ to 0 at $z = h$. Therefore it is given by $s = 1 - \frac{z}{h}$. Hence

$$V = \int_0^h (1 - \frac{z}{h})^2 A dz$$

This integral can be simplified by the change of variables $u = 1 - \frac{z}{h}$, and $du = -\frac{1}{h}dz$, thus

$$V = -Ah \int_{-1}^0 u^2 du = -Ah(\frac{1}{3}u^3)|_{-1}^0 = \boxed{\frac{1}{3}Ah}$$