PROFESSOR: Having looked at the quizzes in a preliminary fashion, I come to the conclusion that some notes on tensors would be useful for the remainder of the term. So at great pain and personal sacrifice I will endeavor to [INAUDIBLE]

All right, let me remind you of where we were a week ago-- before a slight unpleasantness intervened-- and we had just begun to look at the properties of a very useful surface, the representation quadric. And we said that to define it we would take the elements of a second rank tensor, something of the form A11, A12, A13, A21, A22, A23, A31, A32, A33. And we'd use those elements as coefficients in a second rank, a quadratic equation, of the form Aij Xi Xj equals 1. OK, so the Xi and Xj are the coordinates in our three-dimensional space.

And the set of coordinates that satisfy this equation-- setting the right hand side equal to a constant-- will define some surface in space. And it's going to be a surface that is a quadratic form, so it is going to be one of four different surfaces that can be defined by such an equation. One would be an ellipsoid, and in general this would be an ellipsoid oriented in an arbitrary fashion with respect to the reference axes. And it would have a general shape, so the three principle axes of the ellipsoid would be of different lengths.

Then we saw we could also get a hyperboloid of one sheet. One sheet means one surface-- continuous surface. This was the surface that had an hourglass like shape that pinched down in the middle, and the cross-section, in general, would be an ellipsoid. And the asymptotes to this surface would define a boundary between directions in which the radius was real, which meant a positive value of the property, and a direction in which the radius was imaginary, even though that may boggle the mind. And if you take an imaginary quantity and square it, you're going to get a negative number. So this surface would describe a property that was positive in some directions and negative in magnitude over another range of directions.

Then the third surface was a hyperboloid of two sheets. And this was a surface that looked like two [INAUDIBLE] with an elliptical cross-section that were nose to nose, but not touching the origin. And in this range of directions between the asymptote to those two lobes the radius was imaginary, the property negative and then there were a range of directions that would intersect the surface of either of these two sheets in there in those directions the property would be positive.

Then the fourth one-- which is encountered only rarely, but as we pointed out does exist in the case of thermal expansion as a property-- and this would be an imaginary ellipsoid. This would be an ellipsoid in which the distance from the origin out to the surface was everywhere imaginary. And this would be a property then that was negative in all directions in space. And that doesn't seem to be a physically realizable thing, we pointed out that for the linear thermal expansion coefficient there are a small number of very unusual materials that when you heat them up contract uniformly in all directions. Very remarkable property, but that does indeed, thankfully, provide me with an example of an imaginary ellipsoid quadric that does represent, indeed, a realizable physical property.

OK, this surface, we said, had two rather remarkable properties. The first property was that if we looked at the radius from the origin out to the surface of the quadric, a property of the quadric is that the radius out in some direction-- that would be defined in terms of the three direction cosines-- the radius has the property that it is given by 1 over the square root of the value of the property in that direction. Or, alternatively, the value of the property in the direction is 1 over the magnitude of the radius squared. OK, so this is what gives us the meaning of the imaginary radii, the imaginary radii 1 squared would give you a negative property.

But again, I point out, it's obvious here-- but we have to keep it in mind-- that the value of the property and the magnitude of the radius are related in a reciprocal fashion, not only reciprocal, but reciprocal of the square. So if this is the quadric A, a polar quad of the value of the property A as a function of direction would have its maximum value along the minimum principal axis and, correspondingly, the minimum value of the property along the maximum radius of the quadric. So that is sort of counter-intuitive, you think big radius, big value of the property, no, goes not only as the reciprocal, but as the reciprocal of the square. So the value of the property with direction is not a quadratic form any longer, it's based on a quadratic equation, but when you square the radius it's no longer quadratic.

Looks as though we have the value of the property as a function of direction, but looks as though we've lost information about the direction of the resulting vector. The direction that we're talking about is always the direction in which we're applying the generalized force, the electric field, or the temperature gradient, or the magnetic field. But the direction of the thing that happens is defined by the basic equation that gives us the tensor.

But as I demonstrated with you last time, that information is in the quadric also, and this is the so-called radius-normal property. Which doesn't involve exactly what it sounds like, but what it tells us to do is a way of determining the direction of what happens is to go out in a particular direction, along some radius that will intersect the surface of the quadric at some point. And then at the point where the directional vector emerges, construct a vector that is normal to the surface. And so if this then were the direction of an applied electric field, the direction to the-- normal to the surface of the quadric where that direction intersected-- the surface of the representation quadric-- this would be in the case of conductivity the direction in which the current flows.

So everything that you care to know about the anisotropy of the property that's described by a given tensor, and about the direction of what happens-- the generalized displacement as you apply a vector-- is contained within the quadric. But one caveat, this holds only if the tensor is symmetric. And I think this is where we finished up just before the quiz, only if Aij equals Aji does the radius-normal property work. OK, any comments or questions on this?

OK, let me now turn to a practical question of interpretation. If you were indeed to measure a physical property as a function of direction, and you get a tensor that is a general tensor, A11, A12, A13. A21, A22, A23. A31, A32, A33. All the numbers are non-zero. You know that the diagonal values in the tensor are going to give you the value of the property along X1. Or, in general, the diagonal terms give you the value of the property along Xi. Aii gives you the property along Xi.

You have not the foggiest idea, if the quadric has a general orientation, what the maximum and minimum values of the property might be, and these might concern you. Given the tensor that you've determined, what are the largest values, what is the largest value of the property, what is the smallest value of the property? And if the crystal has any symmetry these directions might be the direction of symmetry axes in the crystal. So for many reasons you might want to know the maximum and minimum value of the property.

And then for some applications for a real chunk of crystal that you've examined relative to an arbitrary set of axes, you might want to ask how should I orient that crystal so that I can cut a rod from it that has, let's say, the maximum or minimum thermal conductivity. If you're using a piece of ceramic let's say, as a probe into a furnace, you wouldn't want that probe to conduct much heat, so you'd like the minimum thermal expansion coefficient, for example. If you were making a window out of a transparent single crystal, it'd be a very small window, but you might want the smallest thermal conductivity normal to the surface, and therefore you might want to make your single crystal window oriented in a particular direction. So I hope I've convinced you with enough straw questions that we can knock down easily that, yes, this would be an interesting thing to know.

So how can we do this? Let me show you some simple geometry that let's us set up an equation for finding these directions right away. And it'll be based on the following observation, that when the direction of, let's say, the applied electric field is a long one of the principal axes, then and only then is the direction of the generalized displacement exactly parallel to the radius vector. We go off in any other direction other than a principal axis, the generalized displacement is not parallel to the radius vector out to the surface of the quadric.

We look at the equation of the property A11 X1 plus A12 X2 plus A13 X3. This is the X1 component of the generalized displacement. Similarly, A21 X1 plus A22 X2 plus

A23 X3 is going to be the X2 component. And A31 X1 plus A32 X2 plus A33 times X3, guess what? That's going to be the X3 component of the displacement.

Now we said that these X's give us the direction of the displacement as we let the coordinates of that point X1, X2, X3 roam around the surface of the quadric. When we land at a point on the surface of the quadric which is where a principal axis emerges, then, and only then, is the resulting vector parallel to the applied vector. And this means each component of the resulting vector has to be proportional to a vector out to the point at that location, X1, X2, X3.

So what I'm saying then is that when we are on a principal axis the X1 component of what happens is going to be proportional to X1. And the X2 component of the vector that happens has to be proportional to X2, and this is the radial vector out to that point. And, similarly, the X3 component of what happens has to be parallel to the coordinate X3, the vector out to the surface of the quadric along X3.

So let me make this an equation, now, by putting in a constant for the unknown proportionality constant. And what I'll do, just for arbitrary reasons, is call that proportionality constant "lambda." So this, then, is a set of equations that will give me, if I solve for X1, X2, and X3, the coordinates of one of the three points that sit out on the surface of the quadric at the location where a principal axis emerges.

So let me rearrange these equations to a form that I can solve. And I'll make the equations homogeneous by bringing lambda X1 over to the left hand side of the equation, and let me point out that this is the same lambda in all three equations. There's a proportionality constant between the radial vector out to the surface of the quadric and each component of the vector that results.

So my set of equations will be A11 minus lambda X1 plus A12 times X2 plus A13 times X3, and that's now equal to zero on the right. Next equation would be A21 X1-- and notice I'm not combining A12 and A21 into numerically the same constant, representing it by the same quantity, I'm just leaving them be separate and independent at this point. And then the middle term here is A22 minus lambda times X2 plus A23 times X3 equals 0. And the third equation, you can anticipate how that turns out, it's A31 X1 plus A32 times X2 plus A33 minus lambda, and that's equal to zero.

So this, then, is a set of linear equations and they're called homogeneous equations. And that has solved our problem for us, all we have to do is solve for X1, X2, X3. Except that if the nine coefficients Aij are all arbitrary and lambda is a constant this set of equations has only one solution, and that solution is X1 equals X2 equals X3 equals 0, and that indeed will wipe out every one of these lines. And that is not a very interesting solution. The only case in which this is not the only solution is that if the condition that the determinant of the coefficients is equal to zero, then we can find a real set of X1, X2, X3. So the condition that a solution exists is that A11 minus lambda A12 A13, and A21 A22 minus lambda A23, A31 A32 A33 minus lambda, this determinant has to be equal to zero.

All right, so we know how to expand the determinant, we'll do a little number crunching, and I'm not going to write it down explicitly, but we'll have a term A11 minus lambda and this will be among other terms in the expansion A22 minus lambda A23 A32 A33 minus lambda and then there'll be another determinant that involves this term A12 plus its cofactor plus A13 times its cofactor. Unfortunately none of these terms are zero in general, so I'm going to have three terms here. If I expand this term, I'm going to get something in A11 minus lambda A22 minus lambda times A33 minus lambda and a bunch of other terms that I won't bother to write down explicitly.

The only point I want to draw at this particular juncture is to note that this is a third rank equation. In lambda. It's a cubic equation. And what this means is that there are going to be three roots. And lets call them lambda 1, lambda 2, and lambda 3. And that's the way it should be. There are three principle axes, so we should have three values of the constant lambda that we could put into this equation that will let us solve explicitly for the coordinates X1, X2, and X3, which sits on the surface of the quadric where our principal axis pokes out. Three different principal axes, so this should be three solutions to the third rank equation that we've defined here.

OK I think you've all seen this sort of problem before. The lambdas are called characteristic values. Or, on the basis that things sound more impressive in German, these are called the eigenvalues, meaning the same thing. So using nothing more than the properties of the quadric and some linear algebra we have stumbled headlong into an eigenvalue problem, which is a well known sort of mathematical problem associated with a large number of physical situations and a lot to do with physical properties in particular.

OK, now I have a question for you. How we going to solve this silly thing? I know-- well the answer is I poke it into my laptop, and out comes the answer. But that's not satisfying, we should know how to do this if our batteries were dead and we needed the answer in a hurry. Well, I know how to solve a second rank question. If I have equation of the form ax squared plus bx plus c equals 0, I know the solution to that. It's etched indelibly in my memory and it says that x equals minus b plus or minus the square root of b squared minus 4ac all over 2a. Impressed you, didn't I? You know why I remember that? I learned high school algebra from a throwback-- a teacher who was a throwback to the Elizabethan period.

You know how when people go to a remote island they suddenly find an example of a species that everyone thought was extinct? Well, to secondary education [? Ms. Dourier ?] was a species that had long gone extinct, but happened to be preserved at the high school that I attended. [? Ms. Dourier ?] was a very tall woman, ramrod straight, with a pile of snow white hair on the top of her head. She invariably dressed in a long, black skirt, and a white blouse that had puffy sleeves, and a collar with frilly things on the top, and, invariably, a black ribbon around the frilly lace collar top.

Her mode of enlightened education was to have all of the students in the class have a notebook that was open. And one hapless member of the class was called upon to solve a problem in real time, and as the student recited we all wrote down what the student was saying in our notebooks. All the while [? Ms. Dourier, ?] holding a ruler like a swagger stick, strode up and down the aisles, and woe be unto the poor student who faltered slightly, let alone get something wrong. And that would bring a slap of the ruler down on the desk and a sigh of disgust, and then, all right, you take it, Audrey. And then Audrey would begin to recite until she screwed up and we would all copy in our notebooks.

That tyrant so terrorized and intimidated a bunch of kids in a redneck, working class high school that we really learned algebra. So to this very day, as a reflex action, minus b plus or minus the square root of b squared minus 4ac all over 2a. But I don't have the foggiest idea how to solve a third order equation until I look it up. You probably never had to do it, so let me, for your general education and amusement, pass around a sheet that tells you how to solve a third rank equation.

OK the first step is to convert an equation in, let's say, y squared plus y cubed plus py squared plus qy plus r equals 0 to a so-called normal form where you get rid of one of the terms. So if you make a substitution of variables and let y be replaced by x minus the coefficient p/3, then you get an equation that has a cubic term, linear term x, and a constant. And the constants a and b are combinations of p and q and r in the original equation. And then the solutions, not well known to be sure, are there's a first solution x is equal to a plus b, and then, something messy, x2 and x3 are minus 1/2 of a plus b plus or minus an imaginary term-- those are the second and third roots-- i times the square root of 3 over 2a minus b where A and B are complicated functions of a and b. Capital A and capital B are functions of little a and little b.

And then if the coefficients in the original equation are real, then you get one real and two conjugated imaginary roots. If this combination of terms b squared and a squared are greater than 0, it is b squared over 4 plus a cubed over 27 equals 0. If it's that, then you get three real roots, two of which are equal. In the most general case that we would be encountering here is b squared over 4 plus a cubed over 27 is less than 0, and then there are three real unequal roots.

Unfortunately, if b squared over 4 plus a cubed over 27 is negative that term appears inside of a square root sign in the solutions. So the only case that we'd really be interested in doesn't work for the solution. But fortunately there's another solution and that's given at the bottom of the page, and if you really wanted to solve a third rank equation by hand, these solutions would do it for you. But I think you'd much prefer to have your computer solve the equations for you, but let me show you a way of doing this without any computer, without solving any equations. And it is a very clever method of successive approximations that's based on the properties of the quadric.

So let's suppose we have a tensor, and that tensor has some set of coefficients Aij, all of which are non-zero. And I'll assume although this will work for other quadratic surfaces that the quadric has the shape of an ellipsoid. All right let me now pick some direction at random, actually I don't have to pick it at random, but I'll show you what the shrewd first guess would be.

So let's say we picked this direction. And let us find the direction if this is-- let's do it in terms of current and conductivity. Let's let this be the first guess for the applied field. That's clearly not going to be one of the principal axes, and let this be the first resulting direction of current flow J1. So what I'm finding then is a J sub i in terms of an Aij times an E sub J, and this is my first result, and this was my first assumption.

Let me now let the direction of J be taken as the direction of the applied field for a second guess. And we could normalize to a unit vector, but we don't even have to do that. So let's simply say that my second guess for the electric field that I hope will point along one principal axis is E2, and E2 I'll take as identical-- let's say proportional-- to J1 from my original guess. So this then would be E2, the second guess. And if I find the direction to the normal-- to the quadric in that direction, this would be my second result for the current flow J, I should have put this in parentheses to indicate that these are not components of E and J.

And you can see what's happening, look at where I am, I started out here after just one iteration I have defined this as the potential direction of one of the principal axes. So I'm going to find E1-- the direction of the field-- E1 that comes from sigma ij times Ji, I'll take my second guess E2 either as identical to J1 and this is going to give me then a new value for my second iteration, this would be J2. And this process is going to converge very, very rapidly on the shortest principal axis. As you can see in two shots I'm pretty close to being parallel to this direction, which would be X2 in my illustration.

Now if I wanted to-- if I wanted to see if I was close to convergence-- this is going to be awkward because if I don't normalize the magnitude of the resulting vector, J is going to get larger and larger and larger. So periodically I would want to normalize-- take the magnitude of J, divide that into the components, and then I'd have a unit vector. If I wrote a computer program to do this, I would do the normalization each time to test convergence. Now, this is my kind of procedure, because if I'm doing this by hand and I screw up and I make a mistake and my answer is thrown off a little bit, if I continue to iterate, the thing will proceed to continue to converge to the shortest principal axis. So I can make a mistake and it will correct itself and come back again, and that's my kind of solution.

Yeah, Jason?

AUDIENCE: It's going to give you the minimum value of the property, right?

PROFESSOR: No. It's going to give you the minimum principal axis, that's going to be the maximum value of the property. So that's one out of three, hey, that's not bad. What do I do for the others? Let me tell you without proof to find the maximum principal axis. And that would be the minimum value of the property.

What you would do is exactly the same procedure, and you would operate on not the tensor, but on-- using as a matrix of coefficients-- the matrix of the tensor inverted. And I assume you know how to find the inverse of a matrix, except that you don't have to even find the inverse, the inverse is going to be a collection of functions of the original elements divided by the determinant. You don't have to worry about normalizing, all that's important is the relative value of these coefficients.

Yes, sir?

AUDIENCE: If your tensor is symmetric, can't you say that they'll just [INAUDIBLE] on to another?

PROFESSOR: That's what you're going to do. But when you know just one, that won't work. The other two are floating around somewhere and you don't know in what direction. So if you do this, then you have two out of the three and now very shrewdly, as you point out, if we have two of the principal axes and the tensor is symmetric, then we can automatically get the direction of the third. If it's not symmetric, then you have to use the set of equations, and you have three principal axes as eigenvectors, as they're called in eigenvalue problems. And they do not have to be orthogonal, but you have the components in a Cartesian coordinate system so you can find the angle between those axes.

OK? Yes, sir.

AUDIENCE: With respect to whether the matrix is symmetrical or is not symmetric, the quadric is always-- has three principal axes at right angles right?

PROFESSOR: Only if the tensor is symmetric. Only if the tensor is symmetric. And let me put your question aside for about five minutes, and I want to take an overview of what we've learned about the geometry of the quadric and the symmetry restrictions that we can impose on the tensor in a formal fashion, and see how they compare and how one allows an interpretation of the other. So I'm going to answer your question, I'd like to wait for about three minutes. Other questions?

OK, again this is all very symbolic, I'm not giving you an explicit answer, I can't do it. But what I can do you-- do for you is, ha ha, do to you. I almost slipped and said that, what I can't do to you, is give you a problem that asks you to solve for the principal axes, for example of a property tensor. I will be merciful and perhaps not have all nine coefficients non-zero, I will have one of them zero, or maybe two of them zero. So again to try this it is very instructive and it will cement what the individual steps that I performed here actually involve.

Now, a question that I thought you were going to ask about principal axes is that in order to do what I did here I am using the radius-normal property, and the radius-normal property only works for a symmetric tensor. So sounds like I'm swindling you, except that if I use this procedure, the radius-normal won't actually be identically parallel to the generalized displacement. But it's not going to be terribly different from it, and I'm going to get something that again will converge towards the shortest principal axis. And once I'm close to the principal axis it is true-- symmetric or not-- that the only three directions before which the generalized displacement is parallel to the radius vector are the principal axes.

But anyway this is a dicey situation if the tensor is not symmetric, and it probably comes as a great relief to learn that there's really only one property tensor of second rank that's known for sure to be non-symmetric. So in most cases-- 99 out of 100, if not more-- we will be dealing with property tensors that are symmetric, so we could use this procedure.

So again this property converges remarkably well, and as I say it has the admirable quality of being self-correcting if you make a mistake, if you're grinding this out by hand. But a computer program that you write that just simply takes the direction of J, normalizes it to a unit vector, applies that as the generalized force, finds a quote "J" as a second iteration, normalizes that, and then you can check to whatever degree of convergence you wish to have in your solution. And it's a simple matter to set up a program to do this.

All right, I think we still have some time, so let me now put everything together. And look at what we've seen of the geometric properties of the quadric, and the tensor arrays that we found for different symmetries. For triclinic crystals, for which the recommended procedure is to leave by the nearest exit and work on something different instead, you would have nine elements altogether that are necessary to define the property. I'll discuss this in terms of tensors that gives you an ellipsoid. The argument is not different for hyperboloids of one or two sheets, and we can't draw imaginary ellipsoids, but an ellipsoid is a much easier thing to draw.

OK, triclinic symmetries, either one or one bar, nine elements in the tensor. No constraints whatsoever on the shape of the quadric or on its orientation relative to our coordinate system. So, let's total up now looking at the quadric how many degrees of freedom there are in this situation. We have three principal axes, so those are three degrees of freedom. The orientation of the quadric is arbitrary, so there are three orientational degrees of freedom. And that comes out to six. Supposedly nine degrees of freedom, what are the other three?

Again, if this is a general tensor, which is non-symmetric, the eigenvectors-- the principal axes that you have to choose to squeeze this thing into a diagonal form-- become non-orthogonal. OK? So you have another three degrees of freedom that specify the mutual directions of the eigenvectors-- the angles between them. OK? The directions in which you have to pick your coordinate system, X1, X2, and X3, that force this thing into a diagonal form is going to involve a coordinate system in which these three angles are not 90 degrees and are fixed if you're going to squeeze this thing into a diagonal form. And that's it. So we add these three interaxial angles in, they're a total of nine degrees of freedom for a general non symmetric tensor.

To convince you that these interaxial angles for the eigenvectors really are variables, let me tell you something that we may prove later as a recreational exercise, or I may give it to you on a problem set, it's not difficult to prove. And that is the result that a symmetric tensor remains symmetric for any arbitrary transformation of axes. That is from one Cartesian coordinate system to another.

Now a very salutary effect of this is that the tensor has nine elements. And if the tensor is symmetric-- if you want to transform that tensor-- you only have to do the off-diagonal terms, because whatever it turns into when you change the axes, these off-diagonal terms are going to be equal to the off-diagonal terms in the new tensor. OK? So this means that only six-- if the tensor was symmetric in one coordinate system, only six elements have to be transformed. Actually, it's better than that, you don't have to do six, you only have to do five. And this is the last diagonal term that can be obtained from the trace of the tensor.

And that would be the trace of the tensor T prime after transformation because A11 prime plus A22 prime plus A33 prime has to be equal to the original trace T, which was A11 plus A22 plus A33 in the original system. So if the tensor is symmetric, you really have to crank through a transformation for only five of the nine terms.

Now what is the relevance of this to what I just said? If we have the tensor diagonalized, to a new form A11, 0, 0, 0, A22 prime, 0, 0, 0, A33 prime, that is a special case of a symmetric tensor. So if you decide to go from the coordinate system that produced this diagonalized form to some other coordinate system, the new tensor that you're going to get is going to be symmetric. But the thing was not originally symmetric, so how can that be?

The answer is you can put it in diagonal form only in a non-Cartesian coordinate system. And, therefore, that is evidence that the reference axes-- the principal axes-- cannot be orthogonal. Otherwise you could take the diagonalized tensor and crank it back to some arbitrary Cartesian coordinate, and suddenly it would go to a symmetric tensor when it was not symmetric to begin with. OK? QED. So you can diagonalize a general tensor only if you take the axes along the eigenvectors, which cannot be orthogonal.

OK, I saw you raise your hand, I wasn't ignoring you. That was it? Good.

AUDIENCE: I wanted to make sure that Cartesian [INAUDIBLE]

PROFESSOR: It's always nice to see the class one step ahead of what you're doing.

All right, let's look at a couple of more crystal systems in the form of tensors in those systems and show that that in fact does correspond to the geometric constraints on the quadric as well. For monoclinic crystals-- and this is symmetry 2, symmetry M, and symmetry 2/M. The form of the tensor when we took the coordinate system along symmetry elements was A11, A12, 0-- terms with a single three vanished-- A21, A22, 0, 0, 0, A33. So this was the case where X3 was along the two-fold axis and or perpendicular to the mirror point.

OK, number of independent variables to describe this property is five. And if we look at this in terms of a constraint and shape in orientation of an ellipsoidal quadric, says that one of the principal axes, if the quadric is to remain invariant, has to be along the two-fold axis and or perpendicular to a plane that contains the mirror plane, if a mirror plane is also present.

So what are the degrees of freedom here? This has to be along one reference axis, x3. And, indeed, this form of the tensor occurred only when the two-fold axes were parallel, 2 X3. And this ellipsoid then was left with one degree of freedom. If this is the direction of the reference axis X2, we have a degree of freedom in-- shouldn't have called these X1 and X2, these need not be the coordinate systems. Looking down on the quadric along the two-fold axis, the section of the quadric perpendicular to the two-fold axis can have a general shape, and it can have one degree of freedom in the orientation of the principal axis relative to the coordinate system.

So we have three parameters to specify the principal axes, since this is a general quadric. We have one degree of freedom in orientation, and that adds up to four, but we said five. So what's going on here? Again this is a question of where the eigenvectors point. If this term is not equal to this term, in order to force the tensor into a diagonal form, you have to pick eigenvectors in the X1, X2 plane, which will force the tensor into a symmetric form. That is, to make this term equal to this term. It's only a pair of axes involved, so there's only one angle between these two eigenvectors, which is a variable. So for a nonsymmetric tensor plus one angle between the eigenvectors. So that comes out, a-ha, five.

And again the way to see that is I will never get the tensor into a diagonal form making all the off-diagonal terms zero when I'm starting with a tensor which is not symmetric, and I know that a symmetric tensor, even in the special case where the off-diagonal terms are zero, is going to go to a symmetric tensor in another coordinate system. So I know that there has to be an additional degree of freedom.

OK, the rest come very, very easily, so let me take just a couple of minutes to finish up this stage of the discussion, and we will shortly go on to something different.

The next step up in symmetry is orthorhombic. And if we pick arc-- and this could be

222, 2MM or 2/M 2/M 2/M. We found that when we took X1, X2, and X3 along two-fold axes that the tensor took a diagonal form A11, 0, 0, 0, A22, 0, 0, 0, A33. That says that the quadric if it's an ellipsoid-- or any other quadratic form-- has all three of its principal axes along the reference system X1, X2, X3. So the only degree of freedoms here-- and things are again starting to set up like supercooled water-- the only degrees of freedom are the three principal axes.

And that's it, the orientation is fixed. And, moreover, the tensor is diagonal in this reference system-- it's going to be symmetric in this reference system, it's going to remain symmetric in any other coordinate system. So the tensor is always symmetric, the eigenvectors then are always orthogonal, even if the coordinate system stays Cartesian. And there are three variables, three degrees of freedom.

And finally two remaining cases can be disposed of quickly. We saw that for an arbitrary theta that is a symmetry element-- so let me not say a symmetry-- for an arbitrary rotation about X3. This was the ingenious little proof in which we transform the tensor by an arbitrary rotation theta about X3. We found that the form of the tensor was A11, A12, 0, and now I get to use tensor notation by calling this term A12 again and not the proper indices A21, so I'll put quotation marks. "A22" was equal to "A11."

So going to an arbitrary rotation theta this must be the form of the tensor we discovered, and this will cover then fourfold axes, three-fold axes, and sixfold axes. These two elements are constrained to have the two values, these two are constrained to have the same values, so the tensor is always symmetric. Symmetric relative to these axes, symmetric in all coordinates, and any choice of coordinate system, then.

And that means that if the quadric has this property, it's going to have one principal axis along X3, it's going to be circular in the plane that contains X1 and X2, so there are only, count them up, there are only three degrees of freedom. In terms of tensor elements, these degrees of freedom are A1, A12-- in the diagonalized form-- and A33. And there are three degrees of freedom in terms of the independent elements as well. So this is for anything that involves rotational symmetry other than 180 degrees.

And, finally, for cubic crystals, if a symmetry operation other than 180 degrees gives us this form of the tensor, two off-diagonal terms are zero, two diagonal terms equal. If we impose this along all three reference axes, then the form of the tensor is going to go to a diagonal form with all diagonal terms equal. The quadric that corresponds to that form is a sphere that says that there's one quantity, one degree of freedom in the form of the tensor. There are zero degrees of orientational degree of freedom.

Thus the spherical quadrics stay spherical for any orientation. So again, one independent element in the tensor, just one degree of freedom in the quadric-- namely its radius-- and, again, the formal constraints that we obtained by symmetry transformations agree with those that are the same degrees of freedom when you want the quadric to coincide with the axes.

Yes?

AUDIENCE: Speaking of the circle and the exponents to play only if A12 equals zero?

PROFESSOR: These two were equal, but non-zero. These two are equal, this one was independent. And that would be for the specific case of a fourfold axis. Now if we put a fourfold axis along X1, that's going to involve these two being equal and the non-zero elements become-- let's see-- along X2 it would be A13 and A23. No. OK, we put the four-fold axis along X2, then this one would be A22, A11, and A13 would have to be equal, and 21 and 12 would have to be zero, 23 and 32 would have to be equal. I think that's the way it'll go, and these three have to be zero.

So when you impose all three constraints, here we've had these two equal, now we have these two equal, put the four-fold access in the next-- in the remaining direction, and again it has to be diagonal, and pairwise all of the off-diagonal terms will be required to be zero.

AUDIENCE: I mean, when you diagonalize your matrix the diagonal term are going to be all different in this case if A12 is different from zero.

PROFESSOR: This is not a final result, this is an intermediate step. So we impose this constraint, plus this constraint, and that's actually going to give us all the qualities we're going to get, but we'd want to do the same thing for a four-fold axis along X3. You put them all together, the fact that these are zero will wipe out all of the off-diagonal terms, and these things-- for another choice of axes, these two would have to be zero. And for another choice of the orientation of the four-fold axis, this and this would have to be equal. And, finally, this and this would have to be equal. So this is what we found, in fact, when we cranked through the symmetry transformations, and this is what we would get when we said that the quadric has to have a shape that conforms to cubic symmetric.

AUDIENCE: My question was in the case of the arbitrary relation of those three. All three degrees of freedom, are they, in fact, the three principal axes?

PROFESSOR: For this case here?

AUDIENCE: Yes.

PROFESSOR: For a symmetric tensor, they are two principal axes. OK? If the tensor is nonsymmetric, then these two don't have to be equal any longer, and they give you the extra degree of freedom. OK?

AUDIENCE: What are the three degrees of freedom in this case?

PROFESSOR: I just want you to know I appreciate your question.

AUDIENCE: May I make a suggestion?

PROFESSOR: Yeah.

AUDIENCE: I think you need two degrees of freedom to specify the first one, and since it's symmetric and since it's orthogonal you'd only need one more to find the second one and then another to divide the third. So that's three degrees of freedom to find three principal axes in an orthogonal system.

PROFESSOR: Not sure I like that, because this has to be the shape of the quadric for an arbitrary rotation angle.

AUDIENCE: Does that angle between X1 and X2-- does it have to be 90 degrees if it's not connected?

PROFESSOR: That's correct, that's correct. OK, let's not tie up the whole audience, let me-- let's talk about this and I'll think about it too. But you make a very, very good point, but let's not settle it in real time. We'll throw everybody out, we'll close the door, and you can come back and see who won the scuffle, OK? OK, so let's take our break, I think you're more than ready for it.