PROFESSOR: All right, we've been slogging our way through derivation of the plane groups. And I think I'll do a few more, because we'll stumble across some major tricks in deriving a subfamily of them.

But to not get lost in the forest because of all the trees, I have a set of notes. They are handwritten because my secretary would resign if she had to fit in all these figures and subscripts and strange symbols. So, they are as neat as I could make them. Sorry to say that, in running them through the Xerox machine in an attempt to get everything on one sheet, some of the last lines got clipped. So I'll run these through again and give you a copy that's minus those truncations.

All right. What we've been doing so far, to have a brief reprise, was to take the symmetries, the 10 two-dimensional plane group symmetries. And they were one-, two-, three-, four-, or sixfold axes, a mirror plane, 2mm, 3m, 4mm, 6mm. So there are 10 of them.

And these are the so-called crystallographic point groups. Crystallographic because we considered only those rotation axes that are compatible with a lattice. And they are point groups because they leave at least one point in space, invariant-- stays there rigidly fixed. And they are groups because the collection of operations that are present satisfies the postulates of the mathematical entity called a group.

We've then are in the process of taking these 10 symmetries and adding them to the 5 two-dimensional lattices. The parallelogram net, the rectangular, the centered rectangular, and the square, and the hexagonal. And clearly, we can't put each of those point groups in every one of the lattices. For example, the lattice has to be square for either 4 or 4mm, so we would attempt to place only those two of the 10 point groups into a square net.

We've gone through quite a few of them. I won't bother to draw the pattern of the symmetry elements. But we put no symmetry at all in the general parallelogram net. And that is plane group P1. Maybe I will draw the figures, after all. P2 was a twofold axis dropped into a net, which had to have no specialized shape, simply because a twofold [INAUDIBLE] axis requires nothing but a lattice row if one translation is combined with it.

The threefold axis could fit into an equilateral net. And there's the threefold axis we added to a lattice point. And we have two additional threefold axes in the centers of those 2 equilateral triangles, which each make up half the cell.

P4 was the square net. We put a fourfold axis at the corner of the cell. We've got another one in the middle. Two folds in the middle of the edges because of the 180-degree rotation that is built into the fourfold axis.

And P6. We put that into a hexagonal net. We've got sixfold axes that we dropped in at the lattice point only, nothing else. There's a 120-degree rotation at the lattice point, and that gives us the threefold axes that are present in P3, as well. There's a 180-degree rotation contained in a sixfold axis, and that gives us the twofold axes in all of the locations of P2.

OK, I haven't drawn in any representative patterns, but let me remind you again. The pattern that is characteristic of every one of the plane groups is just the pattern that the point group that you've placed at the lattice point would produce. And that pattern is, in turn, hung at every lattice point of the two-dimensional cell. So even though the huge number of symmetry elements that's present in the higher symmetry point groups is rather intimidating-- you say, wow, how would one draw a pattern for that-- the pattern is nothing more complicated than the pattern of the symmetry that you've placed at the lattice point. And all of these other symmetry elements arise to express relations between the motifs that you've placed at the initial representative lattice point.

Deriving these, we used one theorem, which it's well to remind you of. And that is that if I have a rotation operation A alpha, follow that by a translation that's perpendicular to the rotation axis, what I get is a new rotation operation is B alpha. And it's located in a very specific location. And that is at a location that's a distance x equal half the magnitude of the translation times the cotangent of alpha over 2.

So this combination term, if nothing else, reminds us that these combination theorems, as I call them, are not equations in symmetry elements. They are equations in individual operations. So, for example, if I combine the fourfold axis with the translation, the 180-degree operation that's present in a fourfold axis puts B pi here. The 90-degree rotation puts B pi over 2 here. And the 270-degree rotation-- which I can define just as well as A minus pi over 2-- puts the operation A minus 2 pi over 2 down here. So it's not an equation in symmetry elements. It's an equation in individual operations.

Then some peculiarities started to arise, which we perhaps might not think of. If we put a mirror plane in a primitive rectangular net, that gave a group that we called Pm. If we combine that with a translation, we need a theorem that says what happens if you combine a mirror reflection operation with a translation that's perpendicular to it.

We sketch that out, and once and for all could decide that it's going to be a new reflection operation that's located halfway along this perpendicular translation. So this reflection operation sigma, combined with this translation, gave us a new reflection operation, sigma prime, halfway along the cell. And, of course, we have this one hanging at a lattice point as well.

And then came the interesting one. When we combined a mirror plane now with the centered rectangular net, we have all of the mirror planes that we have here, because this primitive rectangular lattice is a subgroup of the centered lattice. Then the interesting thing happened when we combined the reflection operation sigma with this translation that went to the center of the cell. And that gave us a transformation that was something we had not encountered before.

We took an object, reflected it in this plane, and then translated it down to a centered lattice point over here-- to give us one that sat here-- and then asked, how did I get from the first one, a right-handed one perhaps, to a second one that's a left-handed one, and then a third left-handed one that sits down here? The answer is that we found there was no way we could specify getting from number 1 to number 3 in a single shot. We had to take two steps to do it. And there was nothing more simple than that.

We had to first translate down by the part of the translation-- let me call it tau-- which is equal to that part of the translation t, which is parallel to the reflection operation. And then we had to reflect across, and that would get us from the first to the third. That was a new sort of operation. We'd indicate its locus by a dashed line to distinguish it from a mirror plane. And it has a translation part and a reflection part. Doesn't matter what order in which we do them. We get to the same location if we reflect first and then slide or slide first and then reflect.

So this was a new operation that I'll represent by sigma tau-- looks sort of like a symbol for reflection, but the subscript reminds us you've got to translate by an amount tau that is parallel to the initial mirror plane. And this gave us a new theorem that a general translation that had a part t perpendicular and a component t parallel, when it followed a reflection operation, was equal to a net effect of reproducing the object by a glide plane, sigma tau prime. It had tau equal to the part of the translation that was parallel to the reflection part of the locus. And it was located always at one-half the perpendicular part of the translation.

So using that theorem and completing the mirror planes that hang at the lattice points, we have a very interesting group that consists of a centered lattice, mirror plane hung at the corner lattice point-- this is also a lattice point, so we automatically get the mirror plane in the middle of the cell-- and then in between, we get this new 2-step operation, the glide plane. And the pattern that's representative of this plane group is, as advertised, just what a mirror plane does. And that is hung at every lattice point of the centered rectangular net.

So the glide plane, this new operation that popped up, does what the new operations did in all of the other preceding groups. It tells you how do you get from the pair that you've hung at the lattice point to these that sit in the center of the cell. They're related by translation, but the relation of one to another of these motifs is by the glide plane. So that is an operation which has arisen in the group. And this, then, is a group that we would call Cm. C for a centered rectangular net, as opposed to the primitive net, which is the one we got immediately before.

Again, I sound like a cracked record sometimes, but let me emphasize the simplicity of these patterns. Again, the pattern consists of what you would get when you hung a motif on one lattice point, and then that is repeated by m, which is the symmetry you've placed at a lattice point. And that's hung at every lattice point of the centered rectangular net. The glide planes just express relations between things that you already have when you've hung the motifs on the lattice point. Any questions after this brief reprieve? Comments? Yes, sir.

AUDIENCE: You called those [INAUDIBLE] glides, that's a sigma tau?

PROFESSOR: Yeah. The relation between this one and this one, I've called sigma. That's the reflection operation. The relation between this one and this one down here would be the operation that has a reflection part and a translation part, tau.

Let me point out something that's worth observing when we start making some more complex additions. We said early on that we only have to consider translations that terminate within the unit cell. Because everything is translationally periodic-- not only the atoms in the motifs but the symmetry elements as well. But the observation that I want to make is that if we put a glide plane in. And let's do that for the direct addition of a glide plane to a lattice point. And having done that, we have the potential of possibly having derived a group that we would call Cg.

OK. This diagonal translation-- this is T1, and this is T2. We might ask, what is the reflect [INAUDIBLE] glide operation that sits at the origin lattice point, followed by T1 plus T2. Well, what does our theorem tell us? It says that we should get a new glide plane. It should be at one-half of the perpendicular part of the translation. And the perpendicular part of the translation is T2. So this is at one-half of T2, and it should have a glide component equal to the parallel part of the translation, and that is T1. T1 plus the original glide component, tau.

What is this? A glide component of half of T1 plus the entire T1. If we ask, does that make sense? Yeah, we would glide down to here and then we would translate down to here. And that would give us sitting at half of a path of T1, a new object that sat here.

Is that a glide operation? Yeah, it is. But it is one that is not really distinct from the glide plane that sits here. Because if we have a glide operation with tau equal to one-half of T1, and if this sits in a lattice, then there's going to be a glide plane has tau equal to 1 plus one-half of T1. And there will have to be another glide operation that consists of 2 plus one-half of T1. And the reason is that everything is periodic at an interval T1.

So the moral that I'm trying to draw here is that one can add or subtract to identify the actual nature of a symmetry. One can add or subtract an integral number of translations. And that permits one to reduce any tau to a sigma tau prime, such that tau prime is always less than the translation that's parallel to the tau. In other words, lop off this translation of an entire T1, this translation of the entire T2, and you have identified the basic nature of the glide operation that sits here as something with a translation that's half of T1.

The translations that move motifs out of the cell may be related by a glide operation that involves an integral number of T1s plus half of T1. But it doesn't change the basic nature of the simplest glide step that's in there. That's a very obscure explanation of probably something that didn't puzzle you in the first place, but it's worth saying when we make some of these additions.

Let's finish this off. We have another translation in here, and that's the translation T1 plus T2 over 2. So, what would we get if we took the glide operation sigma one-half of T1 and followed that by a translation T1 plus T2 over 2. And that operation, again, would be equal to a glide plane sigma prime with the tau equal to the original glide component plus the part of the translation that is parallel to the glide plane. And it'd be located at one-half of the perpendicular part of the translation, which is T2. So it'd be at one-half of one-half of T2.

So this says that the combination of the glide operation with the centered translation is a glide plane with a glide component T1. And it's located at one-quarter of T2. And that, indeed, is what you would do if you reflected and reproduced the object by a glide down to here and then translated by T2 over to here.

OK. No, I'm sorry. We glided and then we added on the parallel part of the translation. So we would end up down here. And if one sits here, there has to be one repeated by T1 up here.

And this is exactly the same thing as a reflection plane that's been introduced. So here is a case where we could subtract off the entire translation T1 and say this is identical to a mirror plane passing through the origin, a pure mirror plane sigma that is at one-quarter of T2.

So, let me clean this up and show you what we have. Completing the operations, we would have a pair of objects related by glide, like this, that is hung at every lattice point. It's also hung at the centered lattice point. And what that is going to give us is a pair of objects that sit like this. And what has come in as a result of those combinations is a mirror plane interleaved between the glide planes.

And this is exactly what we have in Cm, which was also interleaved mirror planes and glide planes with the origin shifted by one-half one-quarter of T2. So this is not in a group. Proceeding logically, we'd take Cm and replace the mirror plane by a glide. When we do that, we have a consistent group. But it turns out to be exactly the same arrangement of symmetry elements and exactly the same pattern as Cm, but with a little nudge over along T2 by one-quarter of that translation. Yes.

AUDIENCE: But in Cm, didn't we have mirror planes going through the lattice points?

PROFESSOR: Yeah. And what I'm saying is here there are glide planes going through, but the lattice point is arbitrary. I can put it anywhere I like. And if I decide to put the lattice point here, that turns it into Cm. OK?

So, let's put the two of them side by side. Cm was this mirror plane, mirror plane, mirror plane glides, and the atoms motifs did this. A lattice point here in the center. And I'll deliberately, to make my case, offset the cell by one-quarter of T1. Now I've got a glide plane here, same as this one. Mirror plane at a quarter of T1. Glide plane here at the centered lattice point. Mirror plane here, glide point T2 away. And here are the lattice points. But the pattern of objects looks like this. Armed in advance on what the thing has to look like. Looks like this. And this is exactly the same pattern of motifs.

OK. So, coming out of this consideration, we have with the rectangular nets Pm, Cm, Pg. But Cg, if we try to construct it, was the same as Cm. And that exhausts the possibilities for a single symmetry plane and the rectangular nets.

OK. Let me move on then to the next step. And I'm going to skip over a threefold axis. And I'm going to look at the square net combined with the other symmetry that would require a net with this dimensionality. And this would be a primitive square lattice plus 4 mm.

We've already done almost all the work. So as we derive the symmetries that are subgroups of these higher symmetries, we've done P4. And that says that if we put a fourfold access at the origin, that fourfold axis, we'll get another one in the center of the cell. And we'll get twofold axes in the midpoints of all the edges of the cell.

And then 4 mm has one kind of mirror plane. Two Ms because there are two kinds of mirror planes. The mirror plane says, hey, if I'm in a lattice, I want to be at right angles to a rectangular or a centered rectangular net. Well, OK. This one is happy, because a square net is a special case of a rectangular net. The two translations are merely equal. So he is happy. Combine that with T2, and we'll get another mirror plane like this, and another mirror plane like this. Fourfold axis is going to rotate that mirror plane, so we'll have mirror planes running this way, and this way as well.

And now we have a different kind of mirror plane that we tried to put in in this location. This mirror plane says, hey, I have to be parallel to the edge of a rectangular net or a centered rectangular net. So if we look at the translations that are parallel to and perpendicular to this translation, lo and behold, this mirror plane is aligned along the edge of a centered rectangular net that has the additional specialization of being a centered square. But that mirror plane now is perfectly happy. And he says, OK, I'll hold my piece. I have the arrangements of translations relative to my orientation that makes me happy.

So I can say now that there is a mirror plane running this way. There has to be another mirror plane 90 degrees away from those orientations. And this, then, is going to be the location of all the mirror planes in that net. And at no time did the chalk ever leave my fingers. Just doing what we said we're doing. Putting in 4 mm, making sure the requirements imposed on the lattice are those that that symmetry element demands. And here we go.

Except that there is now one other combination that we have not considered. Here is a mirror plane. And now this mirror plane is diagonal to our translation T2. And here is a mirror plane. And that mirror plane is diagonal to our translation T1. So what is that going to require?

Here is the mirror plane, that's the operation sigma. And here is our translation, T1, down to here. We have a theorem that says that a reflection followed by a translation that has a general orientation is going to give me a new reflection operation, sigma prime, that's located at one-half the perpendicular part of the translation. And it's going to have a glide component that is equal to one-half of the parallel part of the translation.

So what has that told us? To get from this mirror plane down to here, we've gone this far in a sense that is perpendicular to the mirror plane, so this is T perpendicular. And we have a part of the translation that is parallel to the mirror plane. This is T parallel. So the plane that results is going to be a glide plane in here. It's located at one-half of the perpendicular part of the translation. And that is one-half of one-half of T1 plus T2. And it has a glide component equal to T parallel, which is equal to one-half of T1 plus T2.

So we get a new glide plane in here. And that will require glide planes through similar arguments that go down the diagonals of the cell, like this.

And let me convince you that that, in fact, is an operation that must arise if I place, at the origin of the cell, a set of objects that have 4 mm symmetry. We have one like this, one like this, one like this. Another pair hanging here. Another pair hanging here. And if we do a reflection operation, let's say this pair up to this pair, and then slide it down by the diagonal translation-- and we have the same set, again, hanging down here at the diagonally opposed lattice point. Reflect across, slide down to here.

The way in which you get that is, believe it or not, to reflect across this glide plane. How do we get there? We reflect across. We translate down to here. And the way I do that is by reflecting across this diagonal glide.

That probably has convinced no one. But map it out with your own pattern, and I think you'll agree.

Let me observe, while you're considering that. When we derive groups based on 3m or 6mm, we're going to have other cases where a translation along the edge of the cell is inclined to a mirror plane. And the effect of combining a translation with a mirror plane that's inclined to that translation always has the effect of interposing a glide plane halfway in between the mirror planes that are related by translation. So, let me say that in general, because that will be an observation that'll let us identify glide planes quickly.

So the general resolve is if we have a translation inclined to a mirror plane, which of necessity is repeated by translation, a quite general result is that a glide plane is interleaved always between the two. And they'll be parallel because they're related by translation. So we'll see some more cases where we can immediately state that without further thought.

OK. There is something that we might consider doing. I'd like to put it off, though. Here we start with a mirror plane. Can we replace the mirror plane with a glide plane? The answer is yes, we can. But it's not at all clear if I take a fourfold axis and put a glide plane through it, what this plane has to be. So I'd like to leave this one for now and come back to that.

OK. In the notes, I've tried to do all of these derivations thoroughly and logically. So if this is a bit fast or me waving my hands, saying a glide plane is here, and this, that, and the other thing, when you can't really see what I'm pointing at, I think if you refer to the notes that'll be clear. But yes, you had a question?

AUDIENCE: So you said that this glide plane [INAUDIBLE] one-half T perpendicular. What did you write underneath that? I guess that maybe is where your T and one-half T perpendicular is where the glide plane is.

PROFESSOR: OK. I was just demonstrating that, in fact, for this mirror plane with the diagonal translation, the glide plane would come in at one-half of the perpendicular part of the translation. And the part of the translation that-- we were combining this fourfold access with this translation. So the perpendicular part of the translation is half of the body diagonal. And if a glide plane comes in one-half of the perpendicular part of the translation, that put it parallel to the mirror plane and at one-quarter of the way of the translation.

So this little scrawl here says it one-half of T perpendicular and one-half of one-half of the body diagonal. So it's one-quarter. All this equals one-quarter of T1 plus T2.

OK. Now something unexpected happens. If we would move along, I skipped over 3m. We know what P3 looks like. And that is a hexagonal net, an equilateral net. And we have a threefold axis that we've added to the lattice point. And then we found additional threefold axes in the center of these triangles. And, once again, the pattern that has this symmetry is just a triangle of objects hung at every lattice point.

And now we want to add a mirror plane to that threefold axis. So we have a primitive equilateral net plus the two-dimensional plane point group 3m.

The question is, which way should we orient the mirror plane? Well, a mirror plane says, I want to be along the edge of either a centered rectangular net or a primitive rectangular net. There's nothing about this lattice that looks rectangular. Yeah, there is. I see a couple of people doing this, and they have the right idea.

Here's a lattice point. If we go more than one unit cell in this diagram, here's a lattice point. Here's a lattice point. Here's a lattice point. Here's a lattice point. Go down to here and go over to here. And lo and behold, here is a centered rectangular net, hiding incognito in a hexagonal net. Well, that helps. I can put in the mirror plane.

But let me point out-- this is getting very messy, so I'll redraw it on a smaller scale. This will be worth doing so that it's perfectly obvious. I'm going to draw a number of equilateral nets. The reason I'm drawing more than one cell, I would like to point out that here is one centered rectangular net. And it has its edge perpendicular to the translations in the hexagonal net.

But there is also a rectangular net that goes down like this. That's a centered rectangular net that has its edge along one of edges of the unit cell. So I have two possibilities. One is to draw the rectangular net like this. The other one is to draw the rectangular net like this.

And having confused you thoroughly, let me simply draw this larger cell again. And say that I could, if I wanted to, put the mirror planes in in these directions, along the edges of the cell.

OK. And in this case, if you're convinced that there's a centered rectangular net sitting there. Let me clean this up. And here's one mirror plane. 60 degrees away is another mirror plane. And there'll be another mirror plane at this lattice point doing this.

So now I've got 3m at every one of these threefold axes. And I got all those simply by repeating these mirror planes by translation. But in one case, the mirror planes are perpendicular to the cell edge. This being T1, let's say, and this being T2. In this case, the mirror planes are along the edges of the cell. This being T1, and this being T2.

So, holy mackerel! One point group. Two different space groups, which differ in the way in which the symmetry is oriented relative to the lattice. And if you think back a little bit, this particular net, with this dimensional specialization, was happy with a sixfold access as well, and will be compatible with 6mm.

So, really, these two different orientations for the mirror planes, even though we haven't got there yet, are the two different orientations which are both present simultaneously if we were to put 6mm into this hexagonal net. So, same point group. Same lattice. Two different plane groups that depend on the orientation of the symmetry relative to the lattice.

So there isn't even a one-to-one correspondence between the point groups and the plane groups. The pattern for either of these is, once again, going to be just what the pattern of the point group does. If we start with a motif here and repeat that by the threefold axis and the mirror planes, we would have motifs like this.

And here the translation points out in between the mirror planes. In this case, the mirror planes would repeat the objects in two locations, like this. So, indeed, both patterns have 3mm symmetry. And what's different is the way the translations come out in between those mirror planes.

OK. Now our notation has come up against an impasse. We said that the notation for the plane group should be the symbol for the lattice type, which the initiated know has to be a hexagonal net. The point group that we've added, which is 3m, but now how do we distinguish the two different orientations of the mirror planes?

And the way around that is to actually use three symbols. And the second symbol will be what is perpendicular to the cell edge. And the third symbol will be what's parallel to the cell edge. So this one would be called P3M1. And this one, just to distinguish it, would be called P31M.

So if this weren't bad enough, now you've got almost the same symbol with a permutation of two of the terms in it. In this case, the mirror plane is along the cell edge along the translations. In this case, it's perpendicular.

But we're not quite yet done, because we've got translations that are inclined to mirror planes. And now you can see how prudent I was in saying quite generally, without trying to figure out what is the perpendicular part and what is the parallel part, here's a translation and it's at an angle to a mirror plane. And therefore we get a glide that has to be halfway between these mirror planes. And there'll have to be a glide halfway between these mirror planes, and a glide that's halfway between these mirror planes.

In this case, the mirror planes are this way. Again, a translation that's inclined to the mirror plane. We have to have a glide that is midway between these mirror planes. In this case, they're a little easier to identify. They sort of make a triangular box that surrounds the threefold axis.

So now we see another difference in the plane groups and the reason why they're so very distinct. And that is the rearrangement of the glides relative to the threefold axes are quite different in these two cases. So these are two quite different symmetries. Notice that when we start identifying special locations in these two plane groups, here there is a location only of symmetry 3mm for a point group, 3m for a point group.

Here there are two different locations, one that's symmetry 3m, the other one is just symmetry 3. So this is location of point group 3 and a location of point group 3m. So the sort of special positions that exist in these two plane groups will be very, very different.

One more addition, and then I think we can leave with a light heart because we should be done. What about a primitive hexagonal net combined with 6mm? And if you think I'm going to try to draw that, you're crazy. But I can say quite simply and hopefully convince you that this looks like P31M plus P3M1 right on top of one another, because we've got mirror planes that are 30 degrees apart.

So, as a schematic direction and an invitation to do this at home in your spare time, is to take what P6 looks like. And that's sixfold axes at the lattice points. Threefolds here. Twofolds here. And then, directly on top of this, place the mirrors and glides in this orientation, and the mirrors and glides in this orientation.

So we will have mirror planes coming out like that, and glide planes in between all the parallel mirror planes. As I say, I'm heroic but not foolish. Yes.

AUDIENCE: Can you just explain that derivation again? Your P3 [INAUDIBLE].

PROFESSOR: OK. The object of this additional bit of confusion is to put in the symbol the point group, which is 3m, and then tell you how the mirror plane in a point group is oriented relative to the edges of the cell. And we could call them anything we wanted to. We could call them P3M perpendicular, subscript perpendicular. Or P3M subscript parallel to indicate mirror plane parallel to the cell edge. And mirror plane perpendicular to the cell edge. Or we could call them some obscenity, which would be very descriptive, as well. But this is just a way of keeping track of what's parallel to the cell edge and what's perpendicular to the cell edge.

So, this middle symbol is what's oriented perpendicular to the cell edge. And in this case, this is the mirror plane and that is perpendicular. All the mirror planes are perpendicular to the edges of the cell. And then parallel to the cell edge, there's no symmetry at all. So that's 1. 1 stands for no symmetry. In this case, the mirror plane is along the cell edge, so there's an M in the third position. And perpendicular to the cell edge, there's no symmetry at all-- no symmetry plane at all.

So it's purely arbitrary, but we need some mechanism for distinguishing which one we're talking about. That pretty much works as well as anything. And it's a minor perturbation of what we've done for the additional plane group symbols.

All right. It's time for our midafternoon break. It's a nice place to quit, because you're probably feeling good. We've done this, and we can move on to something new.

Actually, there's one more trick. And unless you were really clever, you wouldn't have thought of it. But there's another small family of plane groups that we can get through one particular device. I shouldn't have told you that, because you'll be less inclined to come back now. But we're almost through. And I think it'll be very clear what these additional possibilities are.