3.35 – Fracture and Fatigue
Problem Set 7 – Solutions
December 4, 2003

1.

We are told that the S-N curve for an elastic material follows the Basquin relationship, i.e.

$$\sigma_a = C \cdot N_f^b$$

Where b is approximately equal to -0.09. Say that the total lifetime of the component is n cycles. We are told that it spends 70% of its life at the endurance limit σ_e, 20 % at 1.1 σ_e, and 10 % at 1.2 σ_e. By definition, the lifetime at the endurance limit is $N_f = 10^7$ cycles so that:

$$\sigma_e = C\left(10^7\right)^b$$

Say that the lifetime at $\sigma_a = 1.1\sigma_e$ is N_1 cycles. N_1 is given by:

$$1.1\sigma_e = C\left(N_1\right)^b$$

Using the definition of σ_e we find that:

$$N_1 = 10^7(1.1)^{(1/b)} = 3.468 \times 10^6$$

Similarly, if the lifetime at $\sigma_a = 1.2\sigma_e$ is N_2, N_2 is given by:

$$N_2 = 10^7(1.2)^{(1/b)} = 1.319 \times 10^6$$

We use the Palmgren-Miner law so that

$$\frac{0.7n}{10^7} + \frac{0.2n}{3.468 \times 10^6} + \frac{0.1n}{1.319 \times 10^6} = 1$$

Solving for n, $n = 4.912 \times 10^6$. We can use the information given (failure occurs after 1/4 cycle) to determine the relationship between σ_e and σ_{TS} (not required for this problem but still interesting . . .)

$$\sigma_{\mathrm{TS}} = C \cdot (1/4)^b = 1.133C$$

$$\sigma_e = C \cdot \left(10^7\right)^b = 0.266\sigma_{\mathrm{TS}}$$

You should not necessarily assume that $\sigma_e \approx 0.35\sigma_{\mathrm{TS}} - 0.50\sigma_{\mathrm{TS}}$. That only applies for *some* materials (some steels and copper alloys).

2.

Explain why the modified Goodman diagram can be re-written in terms of the endurance limit, as

$$\sigma_e = \sigma_e|_{\sigma_m=0}\left\{1 - \frac{\sigma_{\rm m}}{\sigma_{\rm TS}}\right\}$$

where $\sigma_e|_{\sigma_m=0}$ is the endurance limit for zero mean stress cyclic loading.

Solution

The modified Goodman equation states that:

$$\sigma_a = \sigma_a|_{\sigma_m=0}\left\{1 - \frac{\sigma_{\rm m}}{\sigma_{\rm TS}}\right\}$$

What does this equation mean? Say we apply a certain stress with no mean stress (call that stress $\sigma_a|_{\sigma_m=0}$) and the component has a certain lifetime. Say we now have a situation with a mean stress $\sigma_{\rm m}$. This equation tells us the stress σ_a we can apply and have the lifetime be the same as in the case with no mean stress. This applies to any stress σ_a, in particular we may let it be the endurance limit σ_e and then we obtain the desired relationship, i.e

$$\sigma_e = \sigma_e|_{\sigma_m=0}\left\{1 - \frac{\sigma_{\rm m}}{\sigma_{\rm TS}}\right\}$$

Since this a constant life relationship, the lifetime will be 10^7 cycles for both σ_e (with no mean stress applied) and $\sigma_e|_{\sigma_m=0}$ (with mean stress applied), so both stresses (σ_e and $\sigma_e|_{\sigma_m=0}$) do indeed represent the endurance limits.

3.

. A circular cylindrical rod with a uniform cross-sectional area of 20 cm^2 is subjected to a mean axial force of 120 kN. The fatigue strength of the material, $\sigma_a = \sigma_{\text{fs}}$ is 250 MPa after 10^6 cycles of fully reversed loading and $\sigma_{\text{TS}} = 500$ MPa. Using the different procedures discussed in class, estimate the allowable amplitude of force for which the shaft should be designed to withstand at least one million fatigue cycles. State all your assumptions clearly.

Solution

The different expressions we have to assess the influence of mean stresses are:

$$\sigma_a = \sigma_a|_{\sigma_m=0}\left\{1 - \frac{\sigma_m}{\sigma_y}\right\} \text{ (Soderberg)}$$

$$\sigma_a = \sigma_a|_{\sigma_m=0}\left\{1 - \frac{\sigma_m}{\sigma_{\text{TS}}}\right\} \text{ (Modified Goodman)}$$

$$\sigma_a = \sigma_a|_{\sigma_m=0}\left\{1 - \left(\frac{\sigma_m}{\sigma_{\text{TS}}}\right)^2\right\} \text{ (Gerber)}$$

In all cases $\sigma_a|_{\sigma_m=0} = 250$ MPa and $\sigma_m = (120{,}000\ \text{N}/.0020\ \text{m}^2) = 60$ MPa. The Modified Goodman and Gerber criteria can be applied directly to give:

$$\sigma_a = 250\left\{1 - \frac{60}{500}\right\} = 220 \text{ MPa, (Modified Goodman)}$$

$$\sigma_a = 250\left\{1 - \left(\frac{60}{500}\right)^2\right\} = 246.4 \text{ MPa, (Gerber)}$$

Applying the Soderberg criterion requires a bit more thought since it includes the yield strength (which is not given) rather than the tensile strength. Depending on the details of the material behavior, the yield stress σ_{YS} could be the same as the tensile strength σ_{TS} (for very brittle materials) or as low as $\approx 0.5\sigma_{\text{TS}}$ (for very ductile materials). I will assume that $\sigma_{\text{YS}} = 0.7\sigma_{\text{TS}} = 350$ MPa. Thus the Soderberg criterion gives:

$$\sigma_a = 250\left\{1 - \frac{60}{350}\right\} = 207.1 \text{ MPa, (Soderberg)}$$

As you can see, you get significantly different answers depending on the model used. The *Soderberg* gives the most conservative result, while *Gerber* is the least conservative.

4.

$E = 210\ GPa \quad A' = 1000\ MPa \quad \sigma_f' = 1100\ MPa$

$\varepsilon_f' = 1.0 \quad n_F = 0.13 \quad b = -0.08$

$c = -0.63$

SHOT PEENING: RESIDUAL COMP STRESS $\sigma_{RC} = 250\ MPa$

(3.5) $$\frac{\Delta\varepsilon}{2} = \frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2A'}\right)^{1/n_F}$$

(7.1) $$\frac{\Delta\sigma}{2} = \sigma_a = \sigma_F'(2N_F)^b$$

(8.5) $$\frac{\Delta\varepsilon}{2} = \frac{\sigma_F'}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c$$ $\sigma_M = 0$ AS FABRICATED (FULLY REVERSED)

$$\left[\frac{\Delta\varepsilon}{2} = \frac{1100\ MPa}{210000\ MPa}(2N_f)^{-0.08} + 1.0(2N_f)^{-0.63}\right]$$

FOR SHOT PEENED, σ_M NOW $= -250\ MPa$

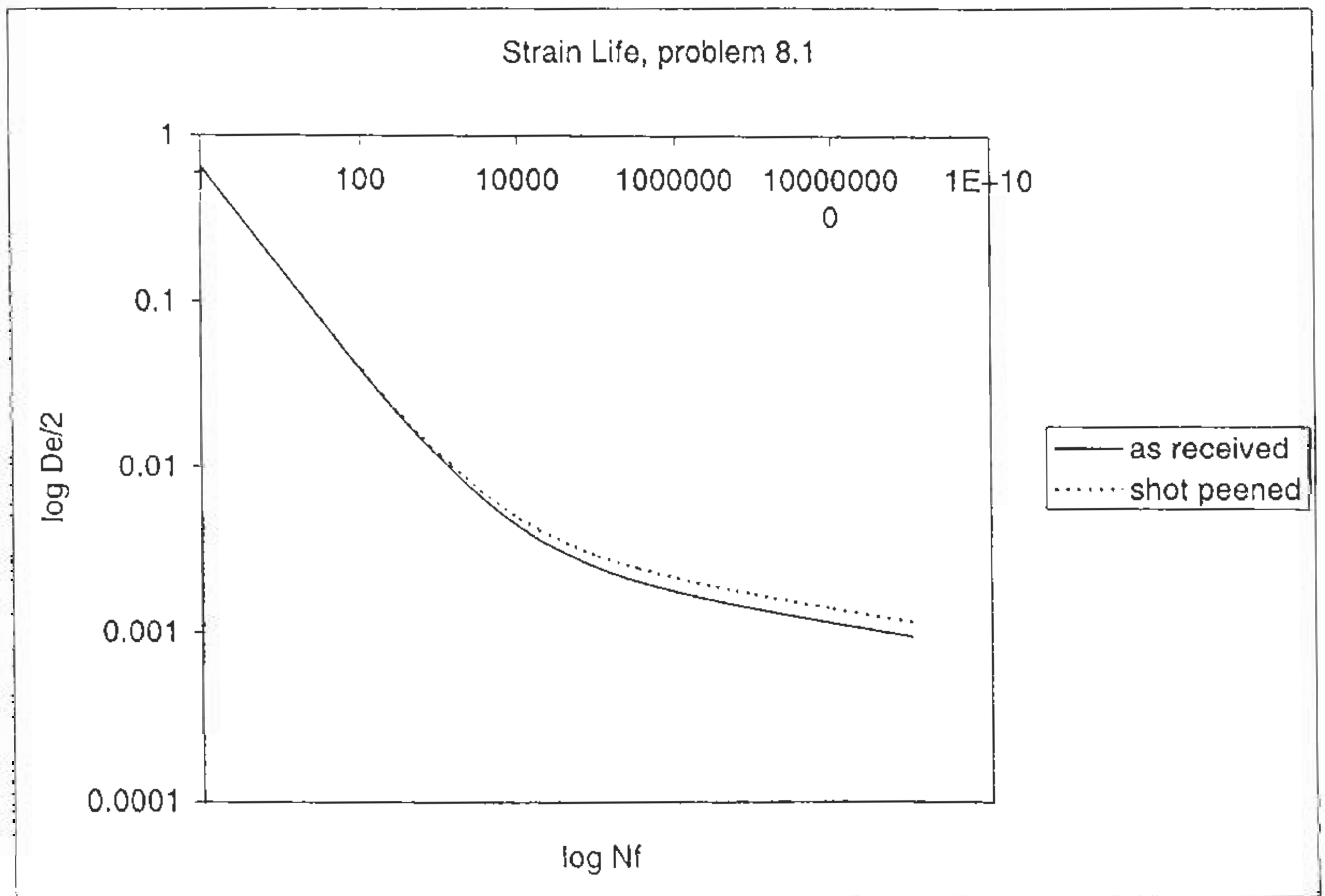
$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_F' - \sigma_M}{E}(2N_f)^b + \varepsilon_F'(2N_F)^c$$

$$\left[\frac{\Delta\varepsilon}{2} = \frac{1350\ MPa}{210000\ MPa}(2N_F)^{-0.08} + 1.0(2N_F)^{-0.63}\right]$$

SO... AS RECEIVED: $$\frac{\Delta\varepsilon}{2} = (5.24\times10^{-3})(2N_f)^{-0.08} + (2N_F)^{-0.63}$$

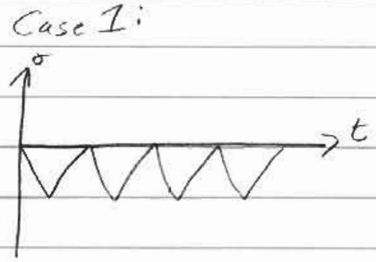
SHOT PEENED: $$\frac{\Delta\varepsilon}{2} = (6.43\times10^{-3})(2N_f)^{-0.08} + (2N_f)^{-0.63}$$

SEE ATTACHED PLOT



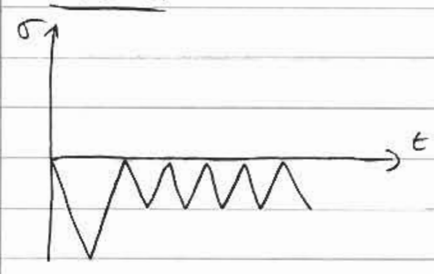
Strain Life, problem 8.1
1
0.1
0.01
0.001
0.0001
100
10000
1000000
10000000
0
1E+10
log De/2
log Nf
as received
shot peened

(5) Case 1:



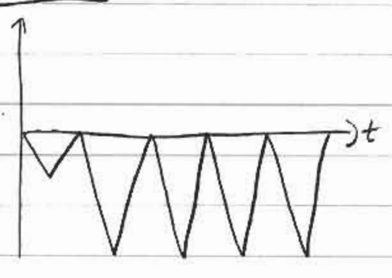
For a non-closing notch, a fatigue crack will initiate in cyclic compression, due to the tensile residual stresses at the notch tip (see notes). The crack will propagate at a progressively decreasing rate, and will subsequently arrest because of crack closure (which becomes more of an issue at longer crack lengths.

Case 2:



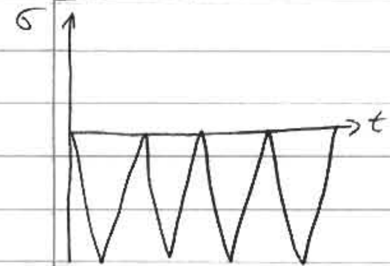
There will be a larger zone of residual tensile stress initially. Therefore, the initial rate of crack growth is higher, and the crack will grow further than in case 1, before it arrests due to crack closure

Case 3:

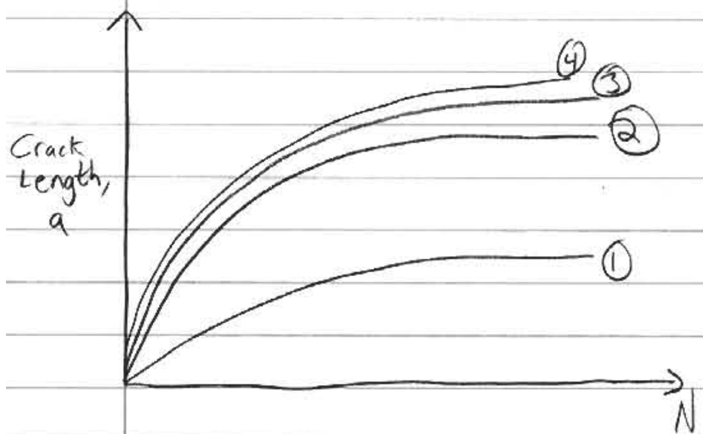


The first cycle could strain harden the material, resulting in a smaller plastic zone than that in Case 4 Assuming the large compressive over loads do not lead to early crack closure, the crack should grow longer than Case 1 and Case 2

Case 4:



It is generally the first cycle that dominates how the crack will grow. This is similar to case 3, but the crack should grow slightly longer due to the first cycle being more strongly compressive (which leads to a larger initial zone of residual tension)



6.

(6) Case I: Tensile overload results in an improvement in fatigue life.

As discussed in class, a tensile overload results in a region of residual compression ahead of a pre-existing crack. Subsequent tensile fatigue cycles must then overcome this residual stress before the crack tip experiences a tensile load. This will improve the fatigue life.

Case II: Tensile overload results in a reduction in fatigue life

It is possible for the tensile over load to initiate new cracks that would not have ordinarily been present during less severe cycling. These new cracks can propagate under the subsequent loading and lead to premature failure. This is why structural components are not intentionally subjected to overloads prior to going into service.

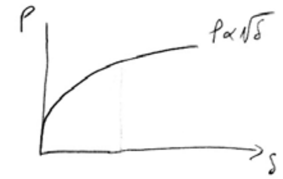
Case III: Tensile overload leads to no change in fatigue life.

An initially defect-free, smooth surface specimen of relatively high strength might be able to sustain an overload without having any cracks nucleate. Assuming the strength is sufficiently high such that the overload did not harden the material, the fatigue life would be roughly the same.

7.

(14.2) Plot the load vs. crack opening displacement curve for a metallic material subjected to loading and unloading phases in zero-tension-zero fatigue:

(a) Plastic deformation at the crack tip; no change in crack configuration (i.e. crack opening and crack length); loading phase.



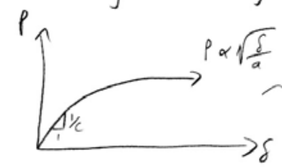
Because there is no change in crack configuration, $\frac{dP}{d\delta} = \infty \rightarrow$ slope is infinite initially. Plastic deformation causes an increase in δ

$P \propto \delta^{1/2}$

(b) Gradual opening of the crack during the loading phase and plastic deformation at the tip:

$$\delta = \frac{K_I^2}{\sigma_y E} \rightsquigarrow \text{Eqn } 9.83$$

$$\delta = \frac{(\sigma\sqrt{\pi a})^2}{\sigma_y E} = \frac{\sigma^2 \pi a}{\sigma_y E} = \frac{\left(\frac{P}{Area}\right)^2 \pi a}{\sigma_y E} \rightarrow \delta \propto P_a^2 \rightarrow P \propto \sqrt{\frac{\delta}{a}}$$

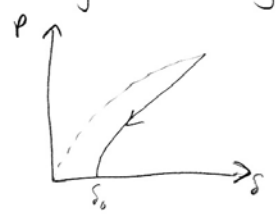


$\rightsquigarrow$ compliance increases and stiffness decreases as crack advances

(c) Elastic behavior at constant crack configuration during the loading phase

$\Rightarrow$ elastic deformation

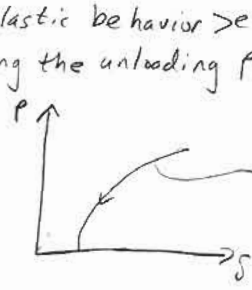
(d) Effect of plastic behavior = effect of configuration change on the P-δ plot during the unloading phase:



$\delta(P=0)$ upon unloading $= \delta_o$

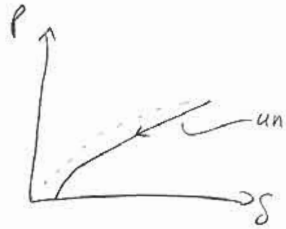
δ_o = difference in δ values between the "saw-cut" and fatigue crack

(e) Effect of plastic behavior > effect of configuration change on the P-δ plot during the unloading phase:



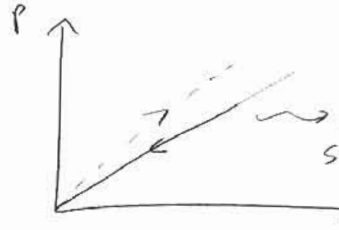
unloading slope is lower than in (d)
↳ plasticity increases compliance

(f) Effect of plastic behavior < effect of configuration change



unloading slope is higher than in (d)
↳ closer to (c)

(g) Configuration change during the unloading, with negligible plastic deformation at the crack tip.



because of crack, compliance ↑, slope decreases