Robinson's Arithmetic

We're developing the idea that a set S is Σ iff it's effectively enumerable iff there is a proof procedure for S. We now want to see that we can we can take the notion of "proof procedure" literally, by treating a proof procedure as a derivation within a certain system of axioms. So we now need to look at systems of axioms.

Definition. *Q*, also known as *Robinson's arithmetic*, is the conjunction of the following axioms:

(Q1) $(\forall x){\sim}\, x = 0$

(Q2) $(\forall x)(\forall y)(sx = sy \rightarrow x = y)$

(Q3) $(\forall x)((x + 0) = x$

(Q4) $(\forall x)(\forall y)(x + sy) = s(x + y)$

(Q5) $(\forall x)(x \bullet 0) = 0$

(Q6) $(\forall x)(\forall y)(x \bullet sy) = ((x \bullet y) + x)$

(Q7) $(\forall x)(xE0) = s0$

(Q8) $(\forall x)(\forall y)(xEsy) = ((xEy) \bullet x)$

(Q9) $(\forall x){\sim}\, x < 0$

(Q10) $(\forall x)(\forall y)(x < sy \leftrightarrow (x < y \lor x = y))$

(Q11) $(\forall x)(\forall y)(x < y \lor (x = y \lor y < x))$

As an account of the natural numbers, Q is pitifully weak. Even the very simplest generalizations, like the commutation laws of addition and multiplication, are underivable in Q. Nevertheless, we have the following:

Theorem. Every true Σ sentence is derivable in Q.

This theorem is why Q is worth looking at. Q is of no interest in itself. Our reason for bringing it up is that it's a single-axiom theory within which every true Σ sentence is provable.

Proof: First, note that, for each m and n,

$0 = [0]$

$s[m] = [sm]$

$([m] + [n]) = [m{+}n]$

$([m] \bullet [n]) = [m{\bullet}n]$

$([m]E[n]) = [mEn]$

are all consequences of Q. An easy induction on the complexity of terms then enables us to prove that, for each closed term τ, there is a number n such the sentence

$\tau = [n]$

is a consequence of Q. An induction shows that each number m has this property:[1]

$(\forall n)(m \neq n \rightarrow Q \vdash \sim [m] = [n]$

A similar induction shows that, for each number n, we have:

For every m, if $m < n$, then $[m] < [n]$ is provable in Q, whereas, if $m \geq n$, then $[m] < [n]$ is refutable[2] in Q.

Thus we see that every atomic sentence is decidable[3] in Q. It follows immediately that every quantifier-free sentence is decidable in Q. Because

$Q \vdash (\forall x)\neg x < 0$

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1 "$\Gamma \vdash \phi$" means that ϕ is a consequence of Γ.

2 A sentence is *refutable* in Q iff its negation is provable in Q.

3 A sentence is *decidable* in Q iff it is either provable or refutable in Q.

and, for each n,

$$Q \vdash (\forall x)(x < [n+1] \leftrightarrow (x = [0] \vee x = [1] \vee \ldots \vee x = [n])),$$

every bounded formula is provably equivalent to an quantifier-free formula. We eliminate bounded quantifiers from he outside in, just as before.

We now see that every bounded sentence is decidable in Q, and so, since Q is true, every true bounded sentence is provable in Q. Consequently, every true Σ sentence can be proven by providing a witness.⊠

Corollary. Let Γ be a true theory that includes[4] Q. Then for each Σ set[5] S, there is a Σ formula that weakly represents S in Γ.

Proof: Let S be the extension of the Σ formula ϕ. If n is in S, ϕ([n]) is a consequence of Q, and so a consequence of Γ. If n ∉ S, ϕ([n]) isn't true, and so it isn't a consequence of Γ.⊠

We can strengthen this corollary by employing a new notion:

Definition. A theory Γ is ω-*inconsistent* iff, for some formula ψ(x), Γ proves (∃x)ψ(x), but it also proves ¬ψ([n]), for each n.

Since an inconsistent theory proves every sentence, every inconsistent theory is ω-inconsistent, but, as we shall see later, not every ω-inconsistent theory is inconsistent. Every true theory is ω-consistent, but not every ω-consistent theory is true.

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4 To say that Γ *includes* Q, in standard usage, it's not literally required that Q be an element of Γ. It's enough that Q is a consequence of Γ. The trouble is that, in standard usage, "theory" is ambiguous between a set of axioms and the set of consequences of the set of axioms. The ambiguous usage is thoroughly entrenched, so we have to live with it.

5 As usual, what we say about sets goes for relations too.

Corollary. Let Γ be an ω-consistent theory that includes[6] Q. Then for each Σ set S, there is a Σ formula that weakly represents S in Γ.

Proof: Let S be the extension of $(\exists y)\psi(x,y)$, where ψ is bounded. The argument that, if n is in S, then $\Gamma \vdash (\exists y)\psi([n],y)$, is the same as above. If n isn't in S, then, for each m, $\psi([n],[m])$ is false, and so $\neg\psi([n],[m])$ is a consequence of Q, and hence a consequence of Γ. It follows by ω-consistency that $(\exists y)\psi([n],y)$ isn't a consequence of Γ.⊠

We cannot strengthen the corollary still further by replacing "ω-consistent" by "consistent," for it is possible to find a consistent theory that includes Q in which not every Σ set is weakly representable. The proof proceeds by starting with a set K that is Σ but not Δ, and by enumerating all the formulas with one free variable. We build up our theory Γ in stages, starting with Q, and at the nth stage adding a sentence to the theory that kills off the possibility that the nth formula weakly represents K, maintaining consistency all the while. I won't go into details.[7]

One can, however, show that, if Γ is a consistent, Σ set of sentences that implies Q, then every Σ set is weakly representable in Γ. The proof requires machinery we haven't developed yet.[8]

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7 You can see the details in a very useful little book by Per Lindstrom entitled *Aspects of Incompleteness* (Springer Verlag *Lecture Notes in Logic,* vol. 10).

8 Again, see Lindstrom's book.

Theorem (Rosser). For any Δ set S, there is a Σ formula that strongly represents S in any consistent theory that includes Q.

Proof: If S is Δ, then there are bounded formulas $\phi(x,y)$ and $\psi(x,y)$ such that $(\exists y)\phi(x,y)$ weakly represents S in Q and $(\exists y)\psi(x,y)$ weakly represents the complement of S. We want to put these formulas together to construct a single formula such that the formula weakly represents S in Q and its negation weakly represents the complement of S. If we were working with true arithmetic rather than Q, we could just take our formula to be $(\exists y)\phi(x,y)$, taking advantage of the fact that $(\forall x)(\sim(\exists y)\phi(x,y) \leftrightarrow (\exists y)\psi(x,y))$ is true. However, we are working with Q, and $(\forall x)(\sim(\exists y)\phi(x,y) \leftrightarrow (\exists y)\psi(x,y))$, though true, might not be provable in Q. So we have to be more devious.

The way our formula $\theta(x)$ is constructed is reminiscent of the way we proved the Reduction Theorem for effectively enumerable sets. There we had effectively enumerable sets A and B, and we wanted to find nonoverlapping effectively enumerable sets $C \subseteq A$ and $D \subseteq B$ with $C \cup D = A \cup B$. The idea was to simultaneously list A and B. If n first turns up in the list for A, put n into C, whereas if n first turns up in the list for B, put it in D; ties go to C. The formula $\theta(x)$ that we're trying to produce describes an analogous construction in which, given n, we simultaneously try to construct a witness to $(\exists y)\phi([n],y)$ and to construct a witness to $(\exists y)\psi([n],y)$. If our first witness is a witness to $(\exists y)\phi([n],y)$, make $\theta([n])$ true, whereas if our first witness is a witness to $(\exists y)\psi([n],y)$, make $\theta([n])$ false; ties go to truth.

The little parable I just told isn't part of the proof. The proof consists in writing down a formula and verifying that its works. The parable was intended to motivate the choice of formula. Here is the formula $\theta(x)$:

$(\exists y)(\phi(x,y) \wedge (\forall z < y)\neg\, \psi(x,y))$.

Let Γ be a consistent theory that includes Q. We need to verify the following four statements:

(a) If n is in S, then $\Gamma \vdash \theta([n])$.

(b) If n isn't in S, then $\Gamma \vdash \neg\theta([n])$.

(c) If n is in S, then $\Gamma \nvdash \neg\theta([n])$.

(d) If n isn't in S, then $\Gamma \nvdash \theta([n])$.

Proof of (a): In n is in S, then $\theta([n])$ is a true Σ sentence, provable in Q and hence in Γ.

Proof of (b): If n isn't in S, then, for some natural number m, $\psi([n],[m])$ is a true bounded sentence, and so a theorem of Q. Consequently,

(1) $(\forall y)([m] < y \rightarrow (\exists z < y)\psi([n],z))$

is a consequence of Q. So are

(2) $(\forall y)([m] < y \rightarrow \sim(\forall z < y)\sim \psi([n],z))$

and

(3) $(\forall y)([m] < y \rightarrow \sim(\phi([n],y) \wedge (\forall z < y)\sim \psi([n],z)))$.

Because n isn't in S, for each k, $\phi([n],[k])$ is false. Consequently, for each k, $\neg(\phi([n],[k]) \wedge (\forall z < [k])\neg\, \psi([n],z))$ is true. Therefore,

(4) $(\forall y)(y < [m] \rightarrow \sim(\phi([n],y) \wedge (\forall z < y)\sim \psi([n],z)))$

is a true bounded sentence, and so a consequence of Q. Also,

(5) $\sim(\phi([n],[m]) \wedge (\forall z < y)\sim \psi([n],z))$

is a true bounded sentence, and so a consequence of Q. (5) is equivalent to

(6) $(\forall y)([m] = y \rightarrow \neg(\phi([n],y) \wedge (\forall z < y)\sim \psi([n],z)))$.

(Q11) gives us this:

(7) $(\forall y)([m] < y \lor ([m] = y \lor y < [m]))$

Combining (3), (4), (6), and (7), we see that

(8) $(\forall y){\sim}(\phi([n],y) \land (\forall z < y){\sim}\, \psi([n],z))$,

which is equivalent to

(9) ${\sim}\theta([n])$,

is a the consequence of Q, and hence a consequence of Γ.

Proof of (c): If n is in S, then, by (a), $\Gamma \vdash \theta([n])$. It follows by consistency that $\Gamma \nvdash {\sim}\theta([n])$.

Proof of (d): If n isn't in S, then by (b), $\Gamma \vdash{\sim}\, \theta([n])$. It follows by consistency that $\Gamma \nvdash$ $\theta([n])$.⊠

Definition. A formula $\sigma(x,y)$ *functionally represents* a total function f in a theory Γ iff, for each n, the sentence $(\forall y)\sigma([n],y) \leftrightarrow y = [f(n)])$ is a consequence of Γ.

Notice that, if our theory Γ (which includes Q) is consistent, any formula that functionally represents f in Γ also strongly represents f in Γ. The converse doesn't hold, in general. If θ strongly represents f in Γ, then, for each m and n,

$(\theta([n],[m]) \leftrightarrow [m] = [f(n)]$

is a consequence of Γ. So we can prove each instance of the generalization:

$(\forall y)(\theta([n],y) \leftrightarrow y = [f(n\_]),$

but there isn't any way to put the proofs of the infinitely many instances together to get a proof of the generalization. So, whereas Rosser's result gives us, for each Δ total function f, a formula that strongly represents f, that formula does not, as a rule, also functionally represent f. However, we can find another formula that does functionally represents f, as we shall now see:

> **Theorem** (Tarski, Mostowski, and Robinson). For any Σ total function f, there is a Σ formula that functionally represents S in any theory that includes Q.

Proof: Since any Σ total function is Δ, Rosser's result tells us that there is a Σ formula θ (x,y) that strongly represents f in Q. Let σ(x,y) be the following formula:

$$(\theta(x,y) \wedge (\forall z < y) \sim \theta(x,z)).$$

The proof that σ functionally represents f in Q (and hence in any theory that includes Q) is a lot like the last proof. Take any n.

If k < f(n), Q $\vdash \sim \theta([n],[k])$, and hence Q $\vdash \sim \sigma([n],[k])$. Also, Q $\vdash \sim [k] = [f(n)]$, and so Q $\vdash (\sigma([n],[k]) \leftrightarrow [k] = [f(n)])$. Since $(\forall y)(y < [f(n)] \rightarrow (\sigma([n],y) \leftrightarrow y = [f(n)]))$ is provably (in Q) equivalent to the conjunction of all the sentences of the form $(\sigma([n],[k]) \leftrightarrow [k] = [f(n)])$ with k < f(n), we see that

(10) $$(\forall y)(y < [f(n)] \rightarrow (\sigma([n],y) \leftrightarrow y = [f(n)]))$$

is a theorem of Q.

Since $(\forall z < [f(m)]) \sim \theta([n],z)$ is provably (in Q) equivalent to the conjunction of all the sentences of the form $\neg\, \theta([n],[k])$, with k < f(n), and since, for each k < f(n), $\sim \theta([n],[k])$ is a consequence of Q, $(\forall z < [f(m)]) \sim \theta([n],z)$ is a consequence of Q. $\theta([n],[f(n])$ is likewise a consequence of Q, so that Q implies $\sigma([n],[f(n)])$, which is logically equivalent to this:

(11) $$(\forall y)(y = [f(n)] \rightarrow (\sigma([n],y) \leftrightarrow y = [f(n)])).$$

Since Q implies $\theta([n],[f(n)]$, it also implies

(12) $(\forall y)([f(n)] < y \rightarrow (\exists z < y)\theta([n],z))$.

(12) is logically equivalent to this:

(13) $(\forall y)([f(n) < y \rightarrow \sim(\forall y)\sim \theta([n],z))$,

which immediately implies this:

(14) $(\forall y)([f(n) < y \rightarrow \sim(\theta([n],y) \wedge (\forall y)\sim \theta([n],y)))$,

that is,

(15) $(\forall y)([f(n)] < y \rightarrow \sim \sigma([n],y))$.

Also, because Q implies

(16) $\sim [f(n)] < [f(n)]$,

Q implies this:

(17) $(\forall y)([f(n)] < y \rightarrow \neg\, y = [f(n)])$.

(15) and (17) together imply this:

(18) $(\forall y)([f(n)] < y \rightarrow (\sigma([n],y) \leftrightarrow y = [f(n)]))$.

(10), (11), (18), and (Q11) together imply:

(19) $(\forall y)(\sigma([n],y) \leftrightarrow y = [f(n)])$.⊠

Robinson's Arithmetic has no intrinsic interrest for us. It's technically useful as a means of proving some theorems, but it's not independently important. In particular, proofs in Q scarcely resemble our intuitive ways of thinking about the natural numbers. We now turn our attention to a much stronger theory, Peano Arithmetic, that does a very good job of reflecting the ways we reason when we prove things informally about the natural numbers.