Subject 24-242. Logic II. Homework due Thursday, April 29

A set A of natural numbers is said to be *m-reducible* (for "many-one reducible") to a set B just in case there is a total Σ function f such that, for any n, n is in A if and only if $f(n)$ is in B.
A is *1-reducible* (for "one-one reducible") to B just in case there is a one-one total Σ function f such that, for any n, n is in A if and only if $f(n)$ is in B.

1. Show that the following are equivalent, for any set A:
   (i) A is recursively enumerable (that is, Σ)
   (ii) A is 1-reducible to the set of Gödel numbers of valid sentences
   (iii) A is m-reducible to the set of Gödel numbers of valid sentences.

   (i) ⇒ (ii). If A is recursively enumerable, then there is a Σ formula $\phi(x)$, with "x" as its only free variable, that weakly represents A in Q. If we set $f(n)$ equal to $\ulcorner$(Q → $\phi([n])^\urcorner$ (where "Q" denotes the conjunction of the axioms of Robinson's arithmetic), then we have $n \in A$ iff Q $\vdash \phi([n])$ iff (Q → $\phi([n])$) is valid iff $f(n) \in$ {Gödel numbers of valid sentences}.

   (ii) → (iii). Trivial.

   (iii) ⇒ (i). Take a one-one total Σ function f such that, for any n, we have $n \in A$ iff $f(n) \in$ {Gödel numbers of valid formulas}; we can find a bounded formula $\phi(x,y,z)$ such that, for any n and m, we have $f(n) = m$ iff $(\exists z)\phi([n],[m],z)$ is true. We know that the set of Gödel numbers of valid formula is Σ, so that there is a bounded formula $\psi(x,y)$ such that, for any m, m is the Gödel number of a valid sentence iff $(\exists y)\psi([m],y)$ is true. Then, for any n, $n \in A$ iff the Σ formula $(\exists y)(\exists z)(\exists w)(\phi([n],y,z) \wedge \psi(y,w))$ is true.

2. Give an example of a Σ partial function that cannot be extended to a Σ total function.

   Let f be the partial function that gives the value 1 if the input is (the Gödel number of) a theorem of Q, the value 0 if the input is a sentence refutable in Q, and is undefined otherwise. f is a Σ partial function, but it cannot be extended to a Σ total function, since if g were such a function, the Σ total function that takes x to max($g(x)$, 1) would the characteristic function of a recursive set that separates the theorems of Q from the sentences refutable in Q.