Subject 24.242. Logic II. HW3 Sample Answers

For each term τ, we have defined a code number $\ulcorner\tau\urcorner$, according to the following prescription:

$\ulcorner 0\urcorner$ = Pair(1,1).
$\ulcorner x_i\urcorner$ = Pair(2,i).
$\ulcorner s\tau\urcorner$ = Pair(4,$\ulcorner\tau\urcorner$)
$\ulcorner(\tau + \rho)\urcorner$ = Pair(5,Pair($\ulcorner\tau\urcorner$,$\ulcorner\rho\urcorner$)).
$\ulcorner(\tau\cdot\rho)\urcorner$ = Pair(6,Pair($\ulcorner\tau\urcorner$,$\ulcorner\rho\urcorner$)).
$\ulcorner(\tau \mathrm{E} \rho)\urcorner$ = Pair(7,Pair($\ulcorner\tau\urcorner$,$\ulcorner\rho\urcorner$)).

Pair(x,y) is, you will recall, $\frac{1}{2}(x + y)(x + y + 1) + x$.

1. Give the Arabic numeral for $\ulcorner(0 + 0)\urcorner$.
Triple(5,$\ulcorner 0\urcorner$,$\ulcorner 0\urcorner$) = Pair(5,Pair($\ulcorner 0\urcorner$,$\ulcorner 0\urcorner$)) = Pair(5,Pair(4,4)) = Pair(5,40) = 1040.

2. Show that a set of natural numbers is decidable if and only if it is either finite or the range of an increasing calculable total function. (A total function f is *increasing* iff, for any x and y, if $x < y$, then $f(x) < f(y)$.)
(⇒) If S is infinite, it is the range of the following increasing total function:

$f(0)$ = the least element of S.
$f(n+1)$ = the least element of S greater than $f(n)$.

If S is decidable, f can be calculated by testing the natural numbers, one after another, for membership in S.

(⇐) A finite set is obviously decidable, just by incorporating a list of the set into the program. If S is the range of an increasing, calculable total function f, we can test whether n is an element of S by calculating $f(0)$, $f(1)$, $f(2)$, and so on, until we reach an i with $f(i) \geq n$. If $f(i) = n$, then n is in S. If $f(i) > n$, then $n \notin S$.