Logic I - Session 10

Plan

- Re: course feedback
- Review of course structure
- Recap of truth-functional completeness?
- Soundness of SD

The course structure

- Basics of arguments and logical notions (deductive validity and soundness, logical truth, falsity, consistency, indeterminacy, equivalence
- SL: syntax and semantics
- Derivation system SD (and SD+)
- Meta-logic: proofs **about** SL and SD / SD+
- PL: syntax and semantics
- Derivation system PD (and PD+, PDE)
- Meta-logic: proofs about PL and PD / PD+ / PDE

Last time

- Mathematical induction
  - Strategy: (1) Insert relevant definitions in the claim you want to prove. (2) Arrange a sequence for the induction. (3) Formulate basis clause and inductive hypothesis. (4) Prove basis clause. (5) Prove inductive hypothesis by assuming its antecedent (n case) and deducing its consequent (n+1 case).
- Truth-functional completeness

Truth-functional completeness

- Truth-function: a mapping, for some positive integer n, from each combination of TVs n sentences can have to a TV.
  - E.g. for two sentences: $\{T,F\} X \{T,F\} \rightarrow \{T,F\}$.
  - More generally: $\{T,F\}^n \rightarrow \{T,F\}$
- SL is truth-functionally complete iff for every truth-function f, there is an SL sentence **P** that expresses f.
  - **P** expresses f iff **P**'s truth-table is the characteristic truth-table for for f

Truth-functional completeness

- We can state this more formally than in TLB:
  - An truth-function f is a set of ordered pairs like this: {**<**<T,T>, T**>**, **<**<T,F>, F**>**, **<**<F,T>, F**>**, **<**<F,F>, F**>**}
  - **P** expresses f iff for any i that is a member of f, when the atomic components of **P** are assigned the TVs in the 1st member of i, **P** receives the TV that's the 2nd member of i.

Truth-functional completeness

- Why care? We want to use SL and truth-tables to test for TF-truth, validity, consistency, etc.
- Suppose we couldn't express some TF in SL, e.g. neither/nor.
- Then we would have no sentence of SL that expressed the same truth-function as 'Neither Alice nor Bill can swim.'
- But then SL wouldn't let us use a TT to show that the sentence is TF-entailed by {`Alice can swim if and only if Bill can swim', `If Alice can swim, then Carol can't swim', `Carol can swim'}.
- Similar points apply to other truth-functions and tests for truth-functional properties and relations

Truth-functional completeness

- So we want to know that we can express **every** truth-function
- We know this because we can set out an algorithm that, for any truth-function f, generates a sentence that expresses f.
- We can do this by focusing on each row of the TT that represents f, finding **characteristic sentences** for each

Truth-functional completeness

- Look at each value left of the vertical line in row i. (We're going to pick a sentence for each value.)
- If the first value is T, we pick A. If it's F, we pick ~A.
- If the second value is T, we pick B. If it's F, we pick ~B, etc.
- Form the iterated conjunction of all these sentences.
- This is the CS for row i.

Truth-functional completeness

- Repeat the procedure for other rows until you have a CS for each row
- Now find a sentence **P** that expresses the TF represented by the whole TT. Look at the values right of the vert line.
  - If there are no Ts, **P** is any contradiction, e.g. A&~A.
  - If there is just one T, on row i, **P** is the CS for row i.
  - If there are Ts on multiple rows, **P** is the iterated disjunction of the CSs for those rows.

- Ex: Find a sentence that expresses the TF for this TT schema:

T	T	T	F	(A&B)&C
T	T	F	F	(A&B)&~C
T	F	T	T	(A&~B)&C
T	F	F	F	(A&~B)&~C
F	T	T	T	(~A&B)&C
F	T	F	F	(~A&B)&~C
F	F	T	T	(~A&~B)&C
F	F	F	F	(~A&~B)&~C

- There are Ts right of the vertical line on rows 3, 5, and 7.
- So we want an interated disjunction of the CSs for those rows.
- (((A&~B)&C) v ((~A&B)&C)) v ((~A&~B)&C)

Soundness

- SD is **sound** iff if $\Gamma \vdash \mathbf{P}$ in SD, then $\Gamma \vDash \mathbf{P}$.
- Why do we care about soundness of SD?
- In doing logic, we care about truth. E.g.: If the sentences in Γ are true, must **P** be true? That is, are derivations always truth-preserving?
- If we want to use a derivation in SD of **P** from Γ to help us tell whether the truth of a given sentence follows from the truth of some other sentences, then derivations in SD better be a guide to truth-functional entailment!
- I.e. it better be that if $\Gamma \vdash \mathbf{P}$ in SD, then $\Gamma \vDash \mathbf{P}$.

Soundness

- So how do we prove that if $\Gamma \vdash$ **P** in SD, then $\Gamma \vDash$ **P**?
- Mathematical induction of course!
- Let's start with a reminder of the definitions for '$\vdash$' and '$\vDash$'.
  - $\Gamma \vDash$ **P** iff every TVA that makes all members of Γ true also makes **P** true.
  - $\Gamma \vdash$ **P** (in SD) iff there is a derivation (in SD) in which all the primary assumptions are members of Γ and **P** occurs in the scope of only those assumptions.

Soundness

- Let's think now about the sequence on which we'll use MI.
- A natural sequence to use is derivation length. We could try:
- Basis clause: For any 1-line derivation (in SD) in which all the primary assumptions are members of Γ and **P** occurs in the scope of only those assumption, $\Gamma \vDash$ **P**.

Soundness

- Then our inductive hypothesis would be:
- IH: If **(A)** For any n-line derivation in which all the primary assumptions are members of Γ^* and **Q** occurs in the scope of only those assumption, $\Gamma^* \vDash$ **Q,** then **(B)** for any n+1-line derivation in which all the primary assumptions are members of $\Gamma^\wedge$ and **R** occurs in the scope of only those assumption, $\Gamma^\wedge \vDash$ **R**.

Soundness

- But this won't work! Exploring why will help us understand why the proof in the book goes the way it does.
- Suppose we've assumed (A), the n-line case.
- Now we're working on (B), the n+1-line case.
    - In this situation, we'd like to be able to know that the nth line of the n+1-line derivation is OK
        - Then we'd just have to show that adding the n+1st line doesn't get us into trouble.
    - So we'd like to use our assumption (A)...
    - But this is where the proof hits trouble...

Soundness

- (A) doesn't guarantee anything about the nth line in an n+1 line derivation! (Why?)
- **(A)** For any n-line derivation in which all the primary assumptions are members of Γ^* and **Q** occurs in the scope of only those assumption, $\Gamma^* \vDash$ **Q.**
- (A) only applies if the sentence on the nth line is only in the scope of primary assumptions!
- And in an n+1 line derivation, the nth line might not be only in the scope of primary assumptions.
- So (A) doesn't guarantee that in our n+1-line derivation we didn't already go wrong in getting to line n.

Soundness

- So what do we do? We need (A) to be stronger, so that it applies to the nth line of an n+1-line derivation. (Compare our proof last time of 6.1E (1a).)
- So we make the inductive hypothesis stronger, and make the basis clause stronger accordingly. That's why the proof in the book is as complex as it is!
- New basis clause: In any derivation, if Γ**1** is the set of open assumptions with scope over sentence **P1** on line 1, then Γ**1** $\vDash$ **P1.**
- Importantly, we're NOT requiring that the assumptions in Γ**1** be **primary** assumptions.

Soundness

- New inductive hypothesis: If (A) then (B).
  - (A) In any derivation, for every line $i \leq n$, if Γi is the set of open assumptions with scope over sentence **Pi** on line i, then $\Gamma i \vDash$ **Pi.**
  - (B) In any derivation, if $\Gamma n+1$ is the set of open assumptions with scope over sentence **Pn+1** on line n+1, then $\Gamma n+1 \vDash$ **Pn+1.**

Soundness

- Now, prove the basis clause:
  - In any derivation, if $\Gamma\mathbf{1}$ is the set of open assumptions with scope over sentence **P1** on line 1, then $\Gamma\mathbf{1} \vDash$ **P1**.
- Since **P1** is on line 1, **P1** must be an assumption.
- And since every assumption is in its own scope, and there aren't any other sentences before **P1**, the set of open assumptions with scope over **P1** is just {**P1**}. So $\Gamma\mathbf{1}$ = {**P1**}.
- Trivially, {**P1**} $\vDash$ **P1**, so since $\Gamma\mathbf{1}$ = {**P1**}, $\Gamma\mathbf{1} \vDash$ **P1**.

Soundness

- Now let's prove the inductive hypothesis by assuming (A).
  - (A) In any derivation, for every line $i \leq n$, if Γi is the set of open assumptions with scope over sentence **Pi** on line i, then $\Gamma i \vDash$ **Pi.**
- Now suppose $\Gamma n+1$ is the set of open assumptions with scope over sentence **Pn+1** on line n+1.
- We need to show that $\Gamma n+1 \vDash$ **Pn+1**.
- (A) entails that $\Gamma n \vDash$ **Pn**, and similarly for every earlier line.
- So we only need to show that we didn't go wrong in the step to line n+1.

Soundness

- **Pn+1** on line n+1 had to be justified by one SD's rules.
- So we can proceed by showing that whichever rule justified **Pn+1**, the result is that Γn+1 ⊨ **Pn+1.**
- I'll just go a couple of the rules. (The other cases are in TLB.)
- Suppose **Pn+1** is justified by conjunction elimination applied to a conjunction **Pn+1&R** (or **R&Pn+1**) on line j.
- We know that since j < n+1, Γj ⊨ **Pn+1&R** (or **R&Pn+1**)
- So Γj ⊨ **Pn+1.**

Soundness

- Now, if **Pn+1** is justified by the sentence on line j, then all the assumptions open at j must still be open at n+1.

- That means that Γj $\subseteq$ Γn+1. So we can show that Γn+1 $\vDash$ **Pn+1** if we can prove the following:

  - (*) If Γj $\vDash$ **Pn+1** and Γj $\subseteq$ Γn+1, then Γn+1 $\vDash$ **Pn+1.**

- We can prove that easily: if a TVA me.m. Γn+1 true and Γj $\subseteq$ Γn+1, then it me.m. Γj is true. So if every TVA that me.m. Γj true also makes **Pn+1** true, then every TVA that me.m. Γn+1 true me.m. Γj true, and hence makes **Pn+1** true.

- So (*) is true. And we know its antecedent is true: Γj $\vDash$ **Pn+1** and Γj $\subseteq$ Γn+1. So it follows that Γn+1 $\vDash$ **Pn+1.**

Soundness

- Now we've made some progress on establishing (B) given (A).
  - (B) In any derivation, if Γn+1 is the set of open assumptions in whose scope is a sentence **Pn+1** on line n+1, then Γn+1 $\vDash$ **Pn+1.**
- For we've shown that given (A), (B) holds whenever **Pn+1** is justified by conjunction elimination.
- If we check all the other rules, then we'll have proven given (A) that **however** we got **Pn+1** from earlier lines, Γn+1 $\vDash$ **Pn+1**.
- So we'll have proven (B) given (A). So we'll have proven the inductive hypothesis and finished our MI proof.

Soundness

- Suppose **Pn+1** is justified by applying ~I to lines h-k ≤ n+1.
- Then **Pn+1** is of the form ~**Q**, line h is **Q**, and lines j ≤ k and k contain some contradictory **R** and ~**R**.
- Since j and k ≤ n+1, we know that (A) applies to lines j and k, so Γj ⊨ **R** and Γk ⊨ ~**R.**
- For ~I to apply, we can't have closed any assumptions between h and n+1 except **Q**. So we know that Γj-**Q** and Γk-**Q** are subsets of Γn+1. So Γj ⊆ Γn+1 ∪ {Q} and Γk ⊆ Γn+1 ∪ {Q}.
- But that means Γn+1 ∪ {Q} ⊨ **R** and Γn+1 ∪ {Q} ⊨ ~**R**.
- So Γn+1 ⊨ ~**Q.**