MONADIC PREDICATE CALCULUS

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1. Introduction and Plato's *The Sophist*

To progress any further, we are going to need an analysis that goes deeper than looking at how complex sentences are formed out of simple sentences. We will have to look at the internal structures of the simple sentences.

A good place to begin is the *Sophist*, where Plato gives an account of what makes the very simplest sentences true or false. Unlike the typical Platonic dialogue, where Socrates plays the dominant role, the principal role in this dialogue is played by Theætetus. Theætetus will go on to distinguish himself as a courageous leader in battle and also as a geometer. It was Theætetus who first discovered the five regular solids—polyhedra all of whose sides and angles are congruent—namely, the cube, the tetrahedron, the octahedron, the dodecahedron, and the icosahedron. But I digress. Here is a quote from Benjamin Jowett's translation:

> *Stranger*. Then, as I was saying, let us first of all obtain a conception of language and opinion, in order that we may have clearer grounds for determining, whether not-being has any concern with them, or whether they are both always true, and neither of them ever false.
>
> *Theætetus*. True.
>
> *Stranger*. Then, now, let us speak of names, as before we were speaking of ideas and letters; for that is the direction in which the answer may be expected.

What they decided about ideas and about names was that some fit together and others do not. For example, you cannot get a word by forming a string of consonants, but you can get a word by combining consonants and vowels in the right way.

> *Theætetus*. And what is the question at issue about names?

Stranger. The question at issue is whether all names may be connected with one another, or none, or only some of them.

Theætetus. Clearly the last is true.

Stranger. I understand you to say that words which have a meaning when in sequence may be connected, but that words which have no meaning when in sequence cannot be connected?

Theætetus. What are you saying?

Stranger. What I thought that you intended when you gave your assent; for there are two sorts of intimation of being which are given by the voice.

Theætetus. What are they?

Stranger. One of them is called nouns, and the other verbs.

Theætetus. Describe them.

Stranger. That which denotes action we call a verb.

Theætetus. True.

Stranger. And the other, which is an articulate mark set on those who do the actions, we call a noun.

Theætetus. Quite true.

Stranger. A succession of nouns only is not a sentence any more than of verbs without nouns.

Theætetus. I do not understand you.

Stranger. I see that when you gave your assent you had something else in your mind. But what I intended to say was that a mere succession of nouns or of verbs is not discourse.

Theætetus. What do you mean?

Stranger. I mean that words like "walks", "runs", "sleeps", or any other words which denote action, however many of them you string together, do not make discourse.

Theætetus. How can they?

Stranger. Or, again, when you say "lion", "stage", "horse", or any other words which denote agents—neither in this way of stringing words together do you attain to discourse; for there is no expression of action or inaction, or of the existence or non-existence indicated by the sounds, until verbs are mingled with nouns; then the words fit, and the smallest combination of them formslanguage, and is the simplest and least form of discourse.

Theætetus. Again I ask, "What do you mean?"

Stranger. When any one says, "A man learns," should you not call this the simplest and least of sentences?

Theætetus. Yes.

Stranger. Yes, for he now arrives at the point of giving an intimation about something which is, or is becoming, or has become, or will be. And he not only names, but he does something, by connecting verbs with nouns; and therefore we say that he discourses, and to this connection of words we give the name of discourse.

Theætetus. True.

Stranger. And as there are some things which fit one another, and other things which do not fit, so there are some vocal signs which do, and others which do not, combine and form discourse.

Theætetus. Quite true.

Stranger. There is another small matter.

Theætetus. What is it?

Stranger. A sentence must and cannot help having a subject.

Theætetus. True.

Stranger. And must be of a certain quality.

Theætetus. Certainly.

Stranger. And now let us mind what we are about.

Theætetus. We must do so.

Stranger. I will repeat a sentence to you in which a thing and an action are combined, by the help of a noun and a verb; and you shall tell me of whom the sentence speaks.

Theætetus. I will, to the best of my power.

Stranger. "Theætetus sits"—not a very long sentence.

Theætetus. Not very.

Stranger. Of whom does the sentence speak, and who is the subject that is what you have to tell.

Theætetus. Of me; I am the subject.

Stranger. Or this sentence, again.

Theætetus. What sentence?

Stranger. "Theætetus, with whom i am now speaking, is flying."

> *Theætetus.* That also is a sentence which will be admitted by every one to speak of me, and to apply to me.
>
> *Stranger.* We agreed that every sentence must necessarily have a certain quality.
>
> *Theætetus.* Yes.
>
> *Stranger.* And what is the quality of each of those two sentences?
>
> *Theætetus.* The one, as I imagine, is false, and the other true.
>
> *Stranger.* The true says what is true about you?
>
> *Theætetus.* Yes.
>
> *Stranger.* And the false says what is other than true?
>
> *Theætetus.* Yes.
>
> *Stranger.* And therefore speaks of things which are not as if they were?'
>
> *Theætetus.* True.
>
> *Stranger.* And say that things are real of you which are not; for, as we were saying, in regard to each thing or person, where is much that is and much that is not.
>
> *Theætetus.* Quite true.
>
> *Stranger.* The second of the two sentences which related to you was first of all an example of the shortest form consistent with our definition.
>
> *Theætetus.* Yes, this was implied in recent admission.
>
> *Stranger.* And, in the second place, it related to a subject?
>
> *Theætetus.* Yes.
>
> *Stranger.* Who must be you, and can be nobody else?
>
> *Theætetus.* Unquestionably.
>
> *Stranger.* And it would be no sentence at all if there were no subject, for as we proved, a sentence which has no subject is impossible.
>
> *Theætetus.* Quite true.
>
> *Stranger.* When other, then, is asserted of you as the same, and not-being as being, such a combination of nouns and verbs is really and truly false discourse.
>
> *Theætetus.* Most true.

In our formal language, *individual constants*, usually lowercase letters from the early part of the alphabet, will play the role of names, and *predicates*, usually uppercase letters, will play the role of verb. Thus "t" will denote Theætetus, and "S" and "F" will represent the actions of sitting and flying, respectively. "Theætetus sits" will be symbolized "St." and "Theætetus flies" will be "Ft." The sentence is true just in case the individual named by the name performs the action designated by the verb.

2. Extension of Plato's *The Sophist*

We want to start with Plato's account and extend it, as far as we can, beyond the very simple sentences Plato considers. The first thing we notice is that simple sentences of the form

name + copula + adjective

or

name + copula + indefinite article + common noun

like

Theætetus is brave.

or

Theætetus is a Greek.

can be readily covered by Plato's account. Thus, we take a simple sentence to consist of a proper name, such as "Theætetus," and a *predicate*, such as "sits" or "is brave" or "is a Greek." The proper name designates an individual and the predicate designates a property or action. The sentence is true just in case the individual has the property or performs the action. We'll symbolize "Theætetus is brave" as "Bt," and we'll use "Gt" to symbolize "Theætetus is a Greek."

We can combine the simple sentences by means of sentential connectives, so that "Theætetus is a brave Greek" will be

$$(Bt \wedge Gt)$$

"Theætetus either sits or flies" will be

$$(St \vee Tt)$$

"Theætetus sits but he does not fly" is

$$(St \wedge \neg Ft)$$

"If Theætetus is brave, so is Socrates" is

$$(Bt \rightarrow Bs)$$

It is tempting to try to treat "Something flies" as analogous to "Theætetus flies." The temptations should be resisted. One way to see that there is a big difference between "Theætetus flies" and "Something flies" is to observe that "Theætetus flies" and "Theætetus is a man" together imply "Theætetus is a man who flies," whereas "Something flies" and "Something is a man" do not imply "Something is a man who flies."

The correct analysis, due to Frege, is this: Whereas "Theætetus flies" and "Theætetus is a man" are to be understood as attributing a property (flying; manhood) to an individual (Theætetus), "Something flies" is to be understood as attributing a property to a property. Namely "Something flies" says about the property of flying that it is instantiated. Similarly, "Something is a man" says about the property of manhood that it is instantiated. We represent the property of flying in English by an open sentence "x flies," and in the formal language by an open sentence "Fx." We indicate that something flies in the formal language by prefixing the *existential quantifier* "$(\exists x)$" to the open sentence "Fx," getting "$(\exists x)Fx$." "$(\exists x)$" is read "for some x" or "there is such an x such that." Similarly, the property of manhood is indicated in English by the open sentence "x is a man" and in the formal language by the open sentence "Mx." We indicate that something is a man by prefixing the existential quantifier to the open sentence "Mx," getting "$(\exists x)Mx$." The property of being a man who flies is indicated in English by the open sentence "x is a man who flies" or "x is a man and x flies" and in the formal language by the open sentence "$(Mx \wedge Fx)$." We indicate that some men fly by prefixing the existential quantifier to the open sentence "$(Mx \wedge Fx)$," getting "$(\exists x)(Mx \wedge Fx)$."

Similarly, it would be tempting to treat "Everything is a man" as analogous to "Theætetus is a man." The resemblance between the two is superficial, however, as we can see from the following example: "Theætetus is either a man or a woman" and "It is not the case that Theætetus is a woman" together imply "Theætetus is a man," whereas "Everything is either a man or a woman" and "It is not the case that everything is a woman" do not imply "Everything is a man." Whereas "Theætetus is a man" indicates that a certain individual (Theætetus) has certain property (manhood), "Everything is a man" attributes a property to a property. Namely, "Everything is a man" tells us about the property of manhood that it is possessed by everything. We indicate that everything is a man by prefixing the *universal quantifier* "$(\forall x)$" (read "for all x" or "for every x") to the open sentence "Mx," getting "$(\forall x)Mx$." We indicate that everything flies by writing "$(\forall x)Fx$." We indicate that all men fly by writing "$(\forall x)(Mx \rightarrow Fx)$," so that, for every x, either x is not a man or else x flies.

3. Use of Venn Diagrams to Illustrate Logical Relationships

3.1. **Quantified Statements.** We can use Venn digrams to illustrate quantified statements.

Example 3.1. "Everyone is a man or a woman" ["$(\forall x)(Mx \vee Wx)$"] is indicated by shading Cell 4 in Figure 1, to indicate that there is nothing in Cell 4.

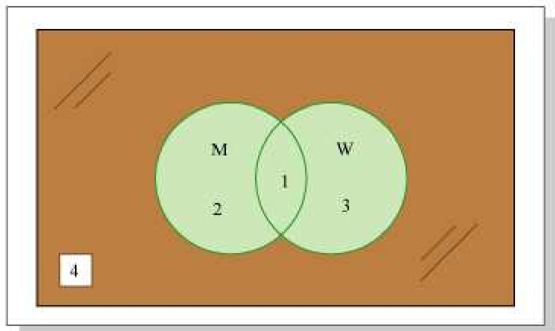

Figure 1. Everyone is a man (M) or a woman (W). "Everyone is a man or a woman" ["$(\forall x)(Mx \vee Wx)$"] is indicated by shading Cell 4, to indicate that there is nothing in Cell 4.

Example 3.2. "All men fly" ["$(x)(Mx \rightarrow Fx)$"] is indicated by shading Cell 2 in Figure 2.

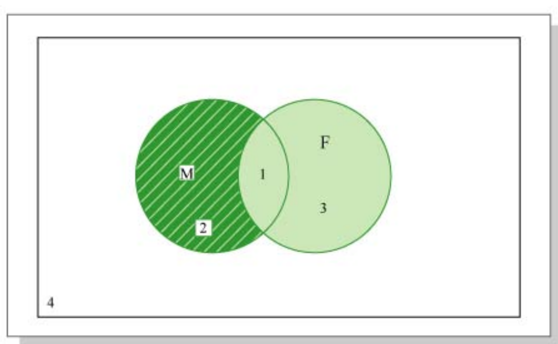

Figure 2. All men (M) fly (F). "All men fly" ["$(x)(Mx \rightarrow Fx)$"] is indicated by shading Cell 2. The shading in Cell 2 means there is nothing in Cell 2.

Example 3.3. "Everything that flies is a man" ["$(\forall x)(Fx \rightarrow Mx)$"] is indicated in Figure 3 by shading Cell 3.

Example 3.4. "Everything that flies is either a man or a woman" ["$(\forall x)(Fx \rightarrow (Mx \vee Wx))$"] is indicated by shading Cell 7 in Figure 4.

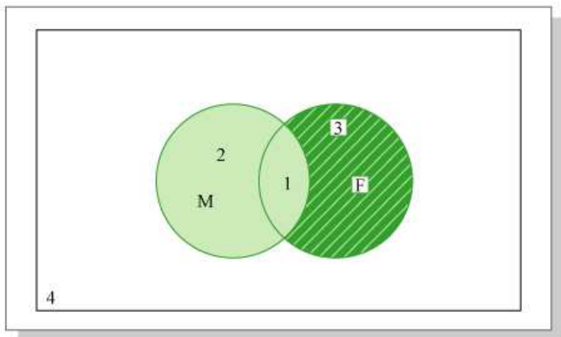

FIGURE 3. Everything that flies (F) is a man (M). "Everything that flies is a man" ["$(\forall x)(Fx \rightarrow Mx)$"] is indicated by shading Cell 3. The shading in Cell 3 means there is nothing in Cell 3.

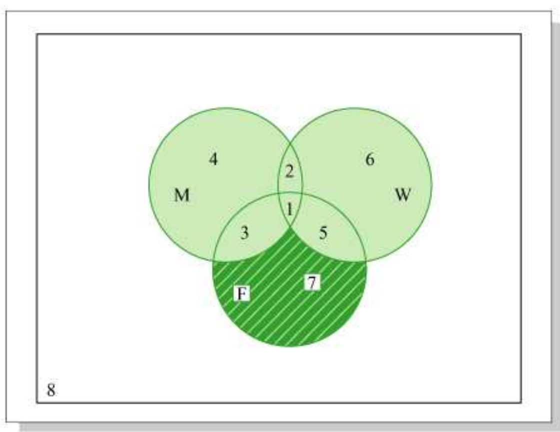

FIGURE 4. Everything that flies (F) is either a man (M) or a woman (W). "Everything that flies is either a man or a woman" ["$(\forall x)(Fx \rightarrow (Mx \vee Wx))$"] is indicated by shading Cell 7. The shading in Cell 7 means there is nothing in Cell 7.

Example 3.5. "Everyone who is either a man or a woman flies" ["$(\forall x)((Mx \vee Wx) \rightarrow Fx)$"] is indicated by shading Cells 2, 4, and 6 in Figure 5.

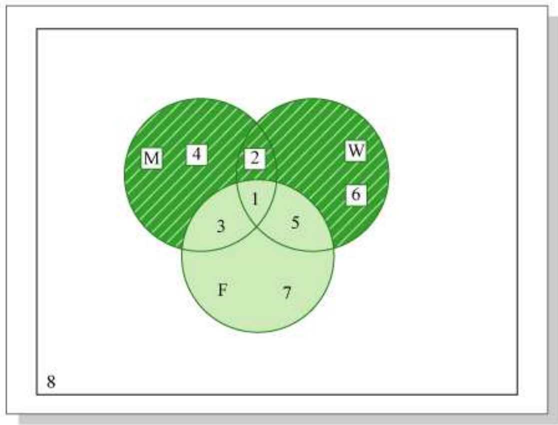

FIGURE 5. Everyone who is either a man (M) or a woman (W) flies (F). "Everyone who is either a man or a woman flies" ["$(\forall x)((Mx \vee Wx) \rightarrow Fx)$"] is indicated by shading Cells 2, 4, and 6. The shading in Cells 2, 4, and 6 means there is nothing in those cells.

Example 3.6. For "Everyone who is both a man and a woman flies" ["$(\forall x)((Mx \wedge Wx) \rightarrow Fx)$"] we shade Cell 2 in Figure 6, while for "Everyone who flies is both a man and a woman" ["$(\forall x)(Fx \rightarrow (Mx \wedge Wx))$"] we shade Cells 3, 5, and 7 in Figure 7.

3.2. **Statements Beginning With an Existential Quantifier.** How about sentences that begin with an existential quantifier? If we want to illustrate the sentence "Someone who is either a man or a woman flies" ["$(\exists x)((Mx \vee Wx) \wedge Fx)$"], we want to indicate that there is something in at least one of the three Cells 1, 3, and 5. We can do this by drawing a curve that passes through Cells, 1, 3, and 5, as in Figure 8. You can think of the curve as like a train track; there is a locomotive somewhere along the track.

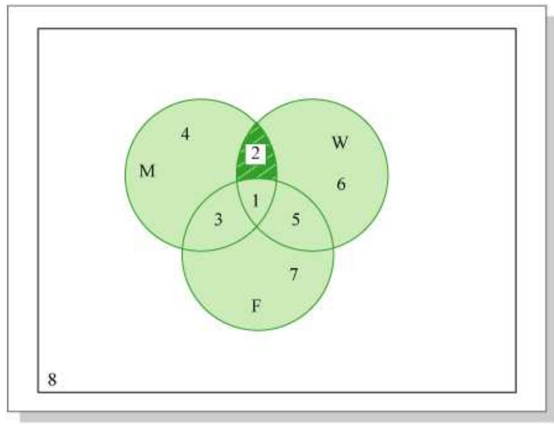

Figure 6. Everyone who is both man (M) and a woman (W) flies (F). For "Everyone who is both a man and a woman flies" ["$(\forall x)((Mx \wedge Wx) \rightarrow Fx)$"], we shade Cell 2. The shading in Cell 2 means there is nothing in Cell 2.

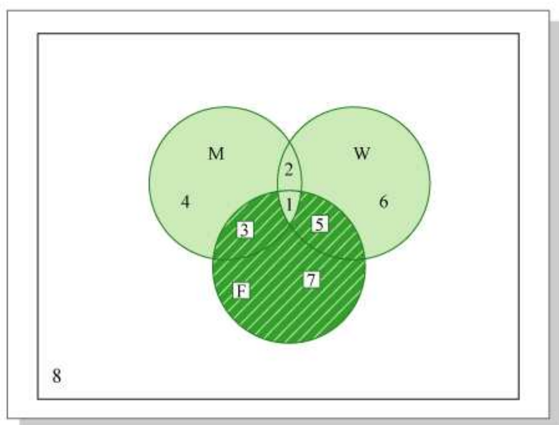

Figure 7. Everyone who flies (F) is both a man (M) and a woman (W). For "Everyone who flies is both a man and a woman" ["$(\forall x)(Fx \rightarrow (Mx \wedge Wx))$"] we shade Cells 3, 5, and 7. The shading in Cells 3, 5, and 7 means there is nothing in those cells.

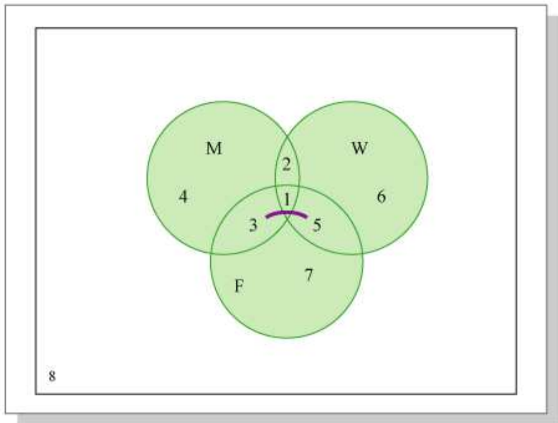

FIGURE 8. Someone who is either a man (M) or a woman (W) flies (F). For "Someone who is either a man or a woman flies ["$(\exists x)((Mx \vee Wx) \wedge Fx)$"] we draw a curve that passes through Cells 1, 3, and 5.

Example 3.7. "There are some men who either sit or fly" ["$(\exists x)(Mx \wedge (Sx \vee Fx))$"] is indicated by a curve that passes through Cells 1, 2, and 3 in Figure 9.

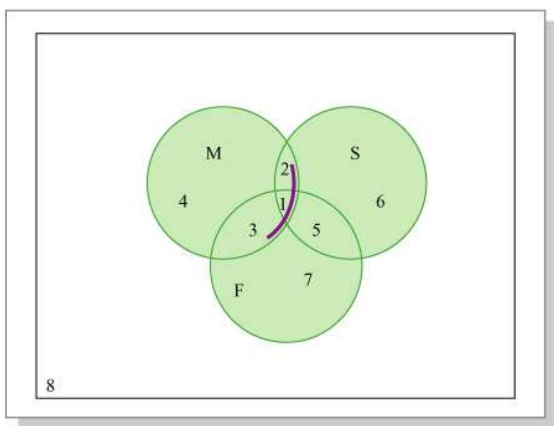

FIGURE 9. There are some men (M) who either sit (S) or fly (F). For "There are some men who either sit or fly" ["$(\exists x)(Mx \wedge (Sx \vee Fx))$"], the curve passes through Cells 1, 2, and 3.

Example 3.8. "There are some men who both sit and fly" ["$(\exists x)(Mx \wedge (Sx \wedge Fx))$"] is indicated by a curve that is contained entirely within Cell 1, as in Figure 10.

Example 3.9. "There are some men who fly, and there are some men who do not" ["$((\exists x)(Mx \wedge Fx) \wedge (\exists x)(Mx \wedge \neg Fx))$"] is indicated in Figure 11 by having a curve that is contained entirely within Cell 2.

Example 3.10. "There are some men who sit, some men who fly, and some men who do neither" ["$((\exists x)(Mx \wedge Sx) \wedge (\exists x)(Mx \wedge Fx) \wedge (\exists x)(Mx \wedge \neg(Sx \vee Fx)))$"] is indicated in Figure 12 by having a curve

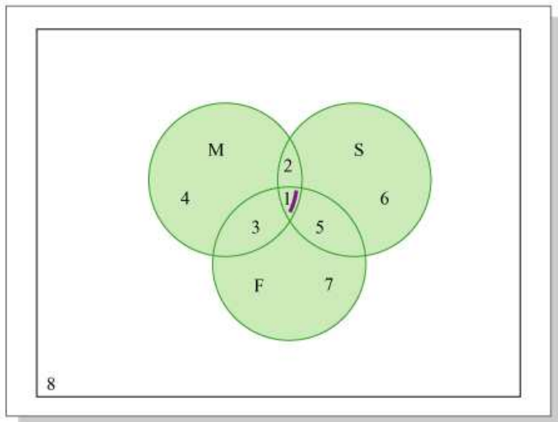

FIGURE 10. There are some men (M) who both sit (S) and fly (F). For "There are some men who both sit and fly" ["$(\exists x)(Mx \wedge (Sx \wedge Fx))$"], the curve is contained entirely within Cell 1.

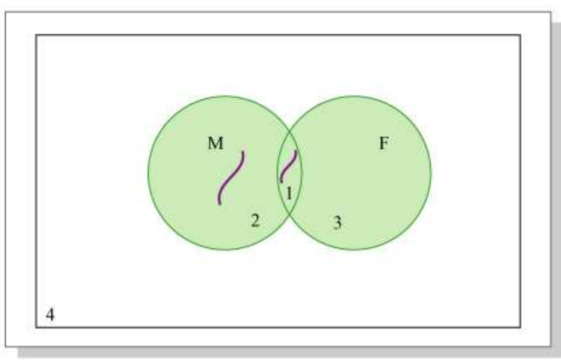

FIGURE 11. There are some men (M) who fly (F), and there are some men who do not. For "There are some men who fly, and there are some men who do not" ["$((\exists x)(Mx \wedge Fx) \wedge (\exists x)(Mx \wedge \neg Fx))$"], one curve is contained entirely in within Cell 1 and another curve is entirely in Cell 2.

that passes through Cells 1 and 2, a second curve that passes through Cells 1 and 3, and yet another curve that is contained entirely within Cell 4.

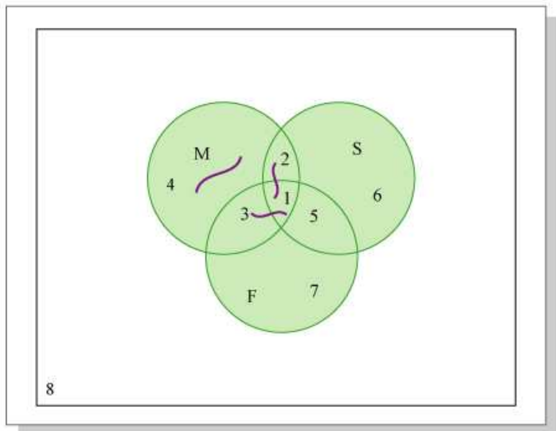

FIGURE 12. There are some men (M) who sit (S), some men who fly (F), and some men who do neither. For "There are some men who sit, some men who fly, and some men who do neither" ["$((\exists x)(Mx \wedge Sx) \wedge (\exists x)(Mx \wedge Fx) \wedge (\exists x)(Mx \wedge \neg(Sx \vee Fx)))$"], there are three curves: one that passes through Cells 1 and 2, one for Cells 1 and 3, and one entirely in Cell 4.

3.3. **Statements Containing Proper Names.** Sentences that contain proper names are indicated the same way, except that we label the curves.

Example 3.11. "Theætetus is a man who either sits or flies" ["$(Mt \wedge (St \vee Ft))$"] is indicated in Figure 13 by having a curve marked "t" pass through cells 1, 2, and 3. Theætetus is a locomotive that is located somewhere along the track.

Example 3.12. "Theætetus is a man who both sits and flies" ["$(Mt \wedge (St \wedge Ft))$"] is indicated in Figure 14 by a curve marked "t" contained entirely within Cell 1.

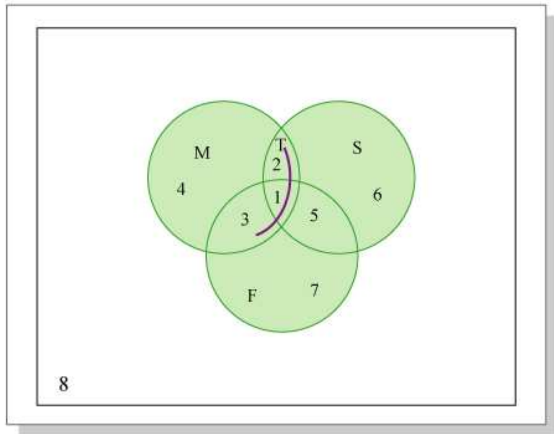

FIGURE 13. Theætetus (t) is a man (M) who either sits (S) or flies (F). For "Theætetus is a man who either sits or flies" ["$(Mt \wedge (St \vee Ft))$"], the curve marked "t" passes through Cells 1, 2, and 3.

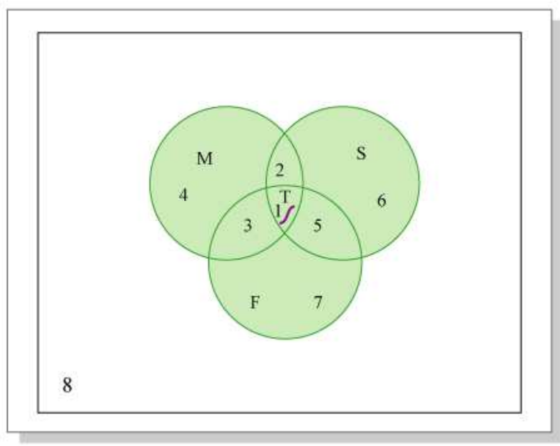

FIGURE 14. Theætetus (t) is a man (M) who both sits (S) and flies (F). For "Theætetus is a man who both sits and flies" ["$(Mt \wedge (St \wedge Ft))$"], the curve marked "$t$" is contained entirely in Cell 1.

Example 3.13. "Rambo, who is a man, does not fly, but Dumbo, who is not a man, does fly" ["$((Mr \wedge \neg Fr) \wedge (\neg Md \wedge Fd))$"] is illustrated in Figure 15 by two curves, one marked "r," contained within Cell 2, and the other, marked "d," contained within Cell 3.

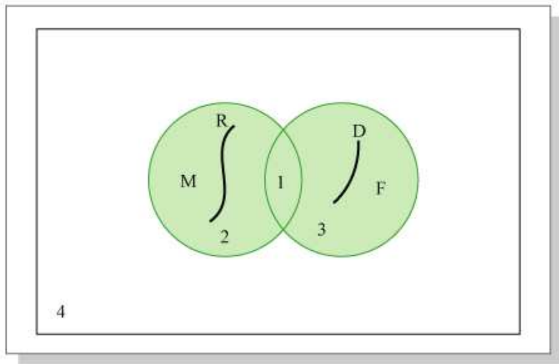

FIGURE 15. Rambo (r), who is a man (M), does not fly (F), but Dumbo (d), who is not a man, does fly. For "Rambo, who is a man, does not fly, but Dumbo, who is not a man, does fly" ["$((Mr \wedge \neg Fr) \wedge (\neg Md \wedge Fd))$"], the curve marked "$r$" is contained within Cell 2 and the curve marked "d" within Cell 3.

3.4. **Validity of Arguments.** We can use Venn diagrams to show that certain arguments are valid.

Example 3.14. For example, consider this argument:

All terriers are dogs.
All dogs are mammals.
Therefore all terriers are mammals.

In symbols,

$(\forall x)(Tx \rightarrow Dx)$
$(\forall x)(Dx \rightarrow Mx)$
$\therefore (\forall x)(Tx \rightarrow Mx)$.

We see whether it is possible to have the premise true and the conclusion false. The first premise is indicated in Figure 16 by shading Cells 3 and 4. The second premise is indicated by shading Cells 2 and 6. If the conclusion were false, there would be something either in Cell 2 or in Cell 4; we indicate this by a train track passing through Cells 2 and 4. But, while the train track would indicate that there is something either within Cell 2 or Cell 4, the fact that Cells 2 and 4 are both shaded indicates that there is nothing in either of those cells. So the attempt to diagram a situation in which the premises are true and the conclusion false ends up with an impossibility. So the argument must be valid.

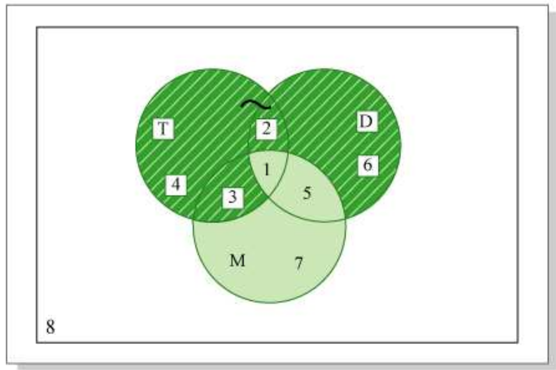

FIGURE 16. All terriers are mammals. All terriers (T) are dogs (D). All dogs are mammals (M). Therefore all terriers are mammals. $(\forall x)(Ex \to Mx)$ $(\exists x)(Ex \wedge Fx)$ $\therefore$ $(\exists x)(Mx \wedge Fx)$. The curve runs through Cells 2 and 4. The shading of cells 2, 3, 4, and 6 indicates there is nothing in those cells.

Example 3.15. Another example:

All elephants are mammals.
Some elephants can fly.
Therefore some mammals can fly.

In symbols,

$(\forall x)(Ex \to Mx)$
$(\exists x)(Ex \wedge Fx)$
$\therefore (\exists x)(Mx \wedge Fx)$.

In Figure 17, we try to diagram a situation in which the premises are true and the conclusion is false. The first premise is indicated by shading Cells 3 and 4. The second premise is indicated by a train track passing through Cells 1 and 3. To say the conclusion is true is to say that there is something either in Cell 1 or in Cell 5. Thus, to indicate that the conclusion is false, we shade Cells 1 and 5. But this has the train track passing entirely through shaded territory, which is impossible. So the argument must be valid.

Example 3.16. Here is an inference to consider:

Dumbo is an elephant.
Dumbo flies.
Therefore some elephants fly.

In symbols,

Ed
Fd
$\therefore (\exists x)(Ex \wedge Fx)$.

To represent the first premise, we draw a train track marked "d" through Cells 1 and 2 in Figure 18. We indicate the second premise by crossing out the part of this train track which lies outside circle "F." To indicate the falsity of the conclusion, we shade Cell 1. But this gives us a train track every part of which is either crossed out or shaded, which represents an impossible situation.

Example 3.17. Now consider this inference:

Traveler is a horse.
All horses eat oats.
Therefore Traveler eats oats.

In symbols,

Ht
$(\forall x)(Hx \rightarrow Ox)$
$\therefore Ot$.

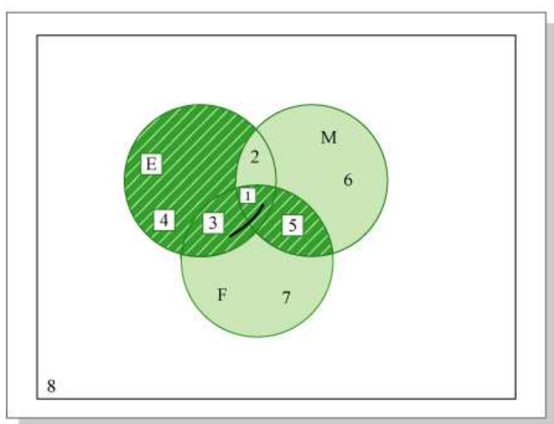

FIGURE 17. Some mammals can fly. All elephants (E) are mammals (M). Some elephants can fly (F). Therefore some mammals can fly. $(\forall x)(Ex \rightarrow Mx)$. $(\exists x)(Ex \wedge Fx)$. $\therefore (\exists x)(Mx \wedge Fx)$. The curve passes through Cells 1 and 3. The shading in Cells 1, 3, 4, and 5 indicates there is nothing in those cells.

The first premise is indicated in Figure 19 by a curve marked "t" passing through Cells 1 and 2, and the second premise is indicated by shading Cell 2. We indicate the falsity of the conclusion by crossing out the part of curve "t" which lies inside circle "O." But this means the whole curve is either crossed out or shaded, which is impossible.

Example 3.18. As a final example, consider

Everyone is either a man or a woman.

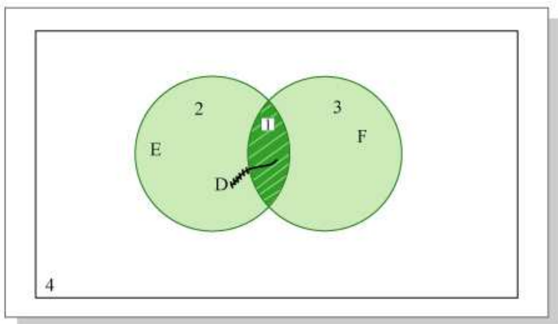

FIGURE 18. Some elephants fly. Dumbo (d) is an elephant (E). Dumbo flies (F). Therefore some elephants fly. Ed. Fd. $\therefore (\exists x)(Ex \wedge Fx)$. The curve marked "$d$" passes through Cells 1 and 2. In Cell 2, the curve is crosshatched. The shading in Cell 1 indicates there is nothing in the cell.

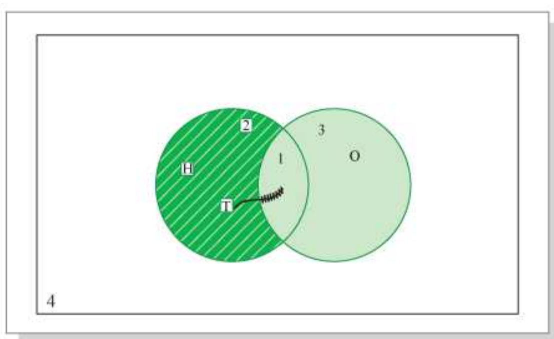

FIGURE 19. Traveler eats oats. Traveler (t) is a horse (H). All horses eat oats (O). Therefore Traveler eats oats. Ht. $(\forall x)(Hx \rightarrow Ox)$. $\therefore Ot$. The curve marked "t" passes through Cells 1 and 2. In Cell 1, the curve is crosshatched; Cell 2 is shaded to indicate nothing in the cell.

Not everyone is a man.
Therefore someone is a woman.

In symbols

$(\forall x)(Mx \vee Wx)$
$\neg(\forall x)Mx$
$\therefore (\exists x)Wx.$

The first premise is indicated in Figure 20 by shading Cell 4. "$(\forall x)Mx$" says that there is notheing in either Cell 3 or Cell 4. "$\neg(\forall x)Mx$" denies this, so it says that there is something either in Cell 3 or in Cell 4, a fact we can indicate by drawing a curve passing through Cells 3 and 4. The conclusion says that there is someone either in Cell 1 or in Cell 3. So we can indicate the falsity of the conclusion by shading Cells 1 and 3. But this has the train track passing entirely through shaded territory, which is impossible.

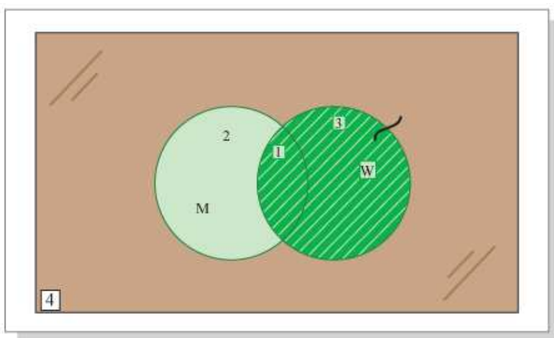

FIGURE 20. Someone is a woman. Everyone is either a man (M) or a woman (W). Not everyone is a man. Therefore someone is a woman. $(\forall x)(Mx \vee Wx)$. $\neg(\forall x)Mx$. $\therefore (\exists x)Wx$. A curve passes through Cells 3 and 4. Cells 1, 3, and 4 are shaded to indicate nothing in the cells.

4. Formal Development

We now turn to a more formal development. A *language for the monadic predicate calculus (MPC)* is given by specifying two kinds of things: *individual constants* (usually lowercase letters from the early part of the alphabet), which play the role of proper names, and *predicates* (usually uppercase letters), which play the roles of intransitive verbs, common nouns, and adjectives. An *atomic formula* consists either of a predicate followed by an individual constant or of a predicate followed by the *variable* "x." The *formulas* of the language constitute the smallest class of expressions which

contains the atomic formulas:
contains $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ whenever it contains ϕ and ψ; and
contain $\neg\phi$, $(\forall x)\phi$, and $(\exists x)\phi$ whenever it contains ϕ.

Proposition 4.1 (Unique Readability)**.** *A formula is built up from atomic formulas in a unique way.*

The subformulas of a particular formula are just the formulas that are contained within the given formula, where a formula is counted as a subformula of itself. If an occurence of the letter "x" within a particular formula is contained within a subformula beginning with "$(\forall x)$" or with "$(\exists x)$," the occurence is said to be bound. Otherwise it is said to be free. A formula with no free occurrences of "x" is a sentence. Where ϕ is a formula and c is a constant, we write $\phi\frac{x}{c}$ for the sentence that results from replacing each free occurrence of "x" in ϕ by "c."

Example 4.2. In "$(Fx \wedge (\forall x)(Gx \wedge \neg(\forall x)Jx))$," the first occurrence of "x" is free, and the other four are bound. "$(Fx \wedge (\forall x)(Gx \wedge \neg(\forall x)Jx))$"$\frac{x}{d}$ is the sentence "$(Fd \wedge (\forall x)(Gx \wedge \neg(\forall x)Jx))$."

Example 4.3. In "$((\forall x)(Fx \leftrightarrow Gx) \wedge ((\exists x)Fx \leftrightarrow (Hx \wedge Jc)))$," the first five occurrences of "x" are bound and the remaining occurrence is free. "$((\forall x)(Fx \leftrightarrow Gx) \wedge ((\exists x)Fx \leftrightarrow (Hx \wedge Jc)))$"$\frac{x}{c}$ is "$((\forall x)(Fx \leftrightarrow Gx) \wedge ((\exists x)Fx \leftrightarrow (Hc \wedge Jc)))$," which is a sentence.

Example 4.4. In "$(((\forall x)Fx \leftrightarrow Gx) \wedge (\exists x)(Fx \leftrightarrow (Hx \wedge Jc)))$," only the third occurrence of "x" is free; the other five are bound. "$(((\forall x)Fx \leftrightarrow Gx) \wedge (\exists x)(Fx \leftrightarrow (Hx \wedge Jc)))$"$\frac{x}{e}$ is the sentence "$(((\forall x)Fx \leftrightarrow Gx) \wedge (\exists x)(Fx \leftrightarrow (He \wedge Jc)))$."

Example 4.5. "Fc" and "$(\forall x)Fx$" are both sentences.

Example 4.6. In general, if ϕ is a formula and c is a constant, $\phi\frac{x}{c}$ is a sentence. Also, every formula which begins with either "$(\forall x)$" or "$(\exists x)$" is a sentence.

Definition 4.7. An *interpretation* (of a language of the MPC) is a function $\mathfrak{A}$ defined on {"$\forall$"} $\cup$ {individual constants of the language} $\cup$ {predicates of the language} that meets the following conditions:

- $\mathfrak{A}$("$\forall$"), also written $|\mathfrak{A}|$, is a nonempty set, called the *universe of discourse* or the *domain* of the interpretation.
- If c is a constant, $\mathfrak{A}(c)$, also written $c^{\mathfrak{A}}$, is an element of $|\mathfrak{A}|$.
- If R is a predicate, $\mathfrak{A}(R)$, also written $R^{\mathfrak{A}}$, is a subset of $|\mathfrak{A}|$.

The universe of discourse of a particular discussion consists of the things we are talking about within that discussion. When I say, sitting at the dinner table with the family, "Everybody who finishes her Brussel sprouts will get ice cream," I'm not promising to reward everyone in the whole world who eats her Brussel sprouts, just everyone sitting there at the table. For any formula ϕ, there will be a set of members of the universe of $\mathfrak{A}$ that satisfy ϕ in $\mathfrak{A}$. If this set is nonempty, the sentence $(\exists x)\phi$ will be true in $\mathfrak{A}$. If every member of $|\mathfrak{A}|$ satisfies ϕ in $\mathfrak{A}$, $((\forall x)\phi))$ will be true in $\mathfrak{A}$. If the member $c^{\mathfrak{A}}$ of $|\mathfrak{A}|$ satisfies ϕ in $\mathfrak{A}$, then the sentence $\phi\frac{x}{c}$ will be true in $\mathfrak{A}$. It makes no sense to talk about a sentence of the formal language being true or false absolutely. A sentence can be either true or false under an interpretration; a formula with free variables cannot.

Intuitively, we have three fundamental semantic notions, truth, falsity, and satisfaction. A sentence expresses a thought that is either true or false, whereas a formula that is not a sentence represents a property, and the formula is satisfied by those elements of the universe that have the property. We shall simplify our treatment by departing from our intuitions a little bit, applying the notion of satisfaction to all formulas, whether or not the formulas contain free variables, stipulating that a true sentence is satisfied by every member of the universe of discourse, whereas a false sentence is satisfied by nothing. Specifically, we have the following:

Proposition 4.8. *Given an interpretation* $\mathfrak{A}$*,*

an atomic formula of the form Rx is satisfied by the members of $\mathfrak{A}(R)$;

an atomic formula of the form Rc is satisfied by every member of the universe if $\mathfrak{A}(c)$ is an element of $\mathfrak{A}(R)$;

otherwise, Rc is satisfied by nothing;

a formula of the form $(\phi \wedge \psi)$ is satisfied by those members of the universe of discourse which either satisfy both ϕ and ψ;

a formula of the form $(\phi \vee \psi)$ is satisfied by those members of the universe of discourse which either satisfy either ϕ and ψ (or both);

a formula of the form $(\phi \rightarrow \psi)$ is satisfied by those members of the universe of discourse which either satisfy ψ or fail to satisfy ϕ;

a formula of the form $(\phi \leftrightarrow \psi)$ is satisfied by those members of the universe of discourse which satisfy both

ϕ and ψ and also by those members of the domain which satisfy neither ϕ nor ψ;

a formula of the form $\neg\phi$ is satisfied by those members of the universe of discourse which fail to satisfy ϕ;

if every member of the universe satisfies ϕ, then every member of the universe satisfies $(\forall x)\phi$;

if some member of the universe fails to satisfy ϕ, nothing satisfies $(\forall x)\phi$;

if some member of the universe satisfies ϕ, every member of the universe satisfies $(\exists x)\phi$;

if no member of the universe satisfies ϕ, no member of the universe satisfies $(\exists x)\phi$;

Example 4.9. As an example of an interpretation, let's let

$$\begin{array}{rcl}
|\mathfrak{A}| & = & \{\text{animals}\} \\
\mathfrak{A}(\text{“}b\text{”}) & = & \text{Bonzo the chimpazee} \\
\mathfrak{A}(\text{“}c\text{”}) & = & \text{Celia the canary} \\
\mathfrak{A}(\text{“}r\text{”}) & = & \text{Reagan, the former president} \\
\mathfrak{A}(\text{“}B\text{”}) & = & \{\text{animals that bay at the moon}\} \\
\mathfrak{A}(\text{“}D\text{”}) & = & \{\text{dogs}\} \\
\mathfrak{A}(\text{“}F\text{”}) & = & \{\text{animals that fly}\} \\
\mathfrak{A}(\text{“}C\text{”}) & = & \{\text{chipmunks}\}
\end{array}$$

Since Celia can fly, $\mathfrak{A}(\text{“}c\text{”})$ is an element of $\mathfrak{A}(\text{“}F\text{”})$, and so every animal will satisfy “Fc”. $\mathfrak{A}(\text{“}r\text{”}) \notin \mathfrak{A}(\text{“}F\text{”})$, since Reagan can't fly, so nothing will satisfy “Fr.” “Bx” will be satisfied by the animals that bay at the moon and “Dx” will be satisfied by the dogs. “$(Dx \wedge Bx)$” will be satisfied by the dogs that bay at the moon. “$\neg Fx$” will be satisfied by the animals that don't fly. “$(Dx \wedge \neg Bx)$” will be satisfied by the dogs that don't bay at the moon. Since some dogs bay at the moon, every animal will satisfy “$(\exists x)(Dx \wedge Bx)$.” Since no dogs fly, nothing will satisfy “$(\exists x)(Dx \wedge Fx)$.” Nothing satisfies “$(\forall x)(Dx \rightarrow Bx)$,” since not every dog bays at the moon. Since Reagan isn't a chipmunk, nothing satisfies “Cr.” So every animal satisfies “$\hat{}\neg Cr$.” So every animal satisfies “$(\forall x)\neg Cr$.” □

Let's introduce some technical jargon. A formula that begins with “$(\forall x)$” is a *universal* formula. One that begins with “$(\exists x)$” is an *existential* formula. Formulas that begin either with “$(\forall x)$” or with “$(\exists x)$” are said to be *initially quantified*. Conjuctions, disjunctions, negations, conditionals, and biconditionals are referred to as *molecular* formulas. Every formula which isn't either atomic or initially quantified is built up from atomic formulas and from initially quantified by means of the connectives “$\wedge$,” “$\vee$,” “$\neg$,” “$\rightarrow$,” and “$\leftrightarrow$.” We refer to those atomic and initially quantified sentences out of which a given sentence is built as its basic truth-functional components.

Definition 4.10. A sentence which is satisfied by every member of $|\mathfrak{A}|$ under an interpretation $\mathfrak{A}$ is said to be true under $\mathfrak{A}$. A sentence which is satisfied by no member of $|\mathfrak{A}|$ under $\mathfrak{A}$ is false under $\mathfrak{A}$.

Theorem 4.11 (Law of Bivalence)**.** *Given an interpretation* $\mathfrak{A}$*, every sentence is either true under* $\mathfrak{A}$ *or false under* $\mathfrak{A}$*.*

Proof. Since every sentence is built up from atomic sentences and from initially quantified sentences by means of the sentential connectives, it will be enough to show that, given an interpretation $\mathfrak{A}$, every atomic sentence and every initially quantified sentence is either true or false under $\mathfrak{A}$ and that every sentence formed from sentences which are either true or false under $\mathfrak{A}$ by means of the sentential connectives is either true or false under $\mathfrak{A}$.

An atomic sentence takes the form "Fc." Such a sentence is true under $\mathfrak{A}$ if $\mathfrak{A}(c) \in \mathfrak{A}(F)$ and false under $\mathfrak{A}$ if $\mathfrak{A}(c) \notin \mathfrak{A}(F)$. A universal sentence $(\forall x)\phi$ is true under $\mathfrak{A}$ if every member of $|\mathfrak{A}|$ satisfies ϕ under $\mathfrak{A}$, and it is false under $\mathfrak{A}$ otherwise. An existential sentence $(\exists x)\phi$ is true under $\mathfrak{A}$ if at least one member of $|\mathfrak{A}|$ satisfies ϕ under $\mathfrak{A}$, and it is false under $\mathfrak{A}$ otherwise.

A conjuction is true under $\mathfrak{A}$ if both conjuncts are true under $\mathfrak{A}$, and it is false under $\mathfrak{A}$ if either conjunct is false under $\mathfrak{A}$. A disconjuction is true under $\mathfrak{A}$ if either disjunct is true under $\mathfrak{A}$, and it is false under $\mathfrak{A}$ if both disjuncts are false under $\mathfrak{A}$. A negation is true under $\mathfrak{A}$ if the negatum is false under $\mathfrak{A}$, and it is false under $\mathfrak{A}$ if the negatum is true under $\mathfrak{A}$. A conditional is true under $\mathfrak{A}$ if the antecedent is false under $\mathfrak{A}$ or the consequent is true under $\mathfrak{A}$; if the antecedent is true under $\mathfrak{A}$ and the consequent is false under $\mathfrak{A}$, the conditional is false under $\mathfrak{A}$. A biconditional is true under $\mathfrak{A}$ if both components are true under $\mathfrak{A}$ or both components are false under $\mathfrak{A}$; if one component is true and the other is false, the biconditional is false under $\mathfrak{A}$. $\square$

Corollary 4.12. *For any sentence* ϕ*, interpretation* $\mathfrak{A}$*, and element* a *of* $|\mathfrak{A}|$*,* ϕ *is true under* $\mathfrak{A}$ *iff* a *satisfies* ϕ *under* $\mathfrak{A}$*.*

Proof. If ϕ is true under $\mathfrak{A}$, then, by defnition of"true," every element of $|\mathfrak{A}'|$ satisfies ϕ under $\mathfrak{A}$. So in particular, a satisfies ϕ under $\mathfrak{A}$. If, on the other hand, ϕ isn't true under $\mathfrak{A}$, then, by bivalence, ϕ is false under $\mathfrak{A}$, so that, by definition of "false," nothing satisfies ϕ under $\mathfrak{A}$; so, in particular, a doesn't satisfy ϕ under $\mathfrak{A}$. $\square$

The following definition is taken over directly from the sentential calculus:

Definition 4.13. A *normal truth assignment (N.T.A)* is a function which assigns a number, either 0 or 1, to each sentence, subject to the following conditions

- A conjuction is assigned 1 iff both conjuncts are assigned 1.
- A disconjuction is assigned 1 iff one or both disconjuncts are assigned 1.
- A negation is assigned 1 iff the negatum is assigned 0.
- A conditional is assigned 1 iff the antecedent is assigned 0 or the consequent is assigned 1.
- A biconditional is assigned 1 iff both components are assigned the same value.

Definition 4.14. A sentence is *tautological* iff it is assigned the value 1 by every N.T.A. A sentence is *valid* iff it is true under every N.T.A

For the sentential calculus, the words "tautological" and "valid" were different words for the same thing. Now that we've started on the predicate calculus, we need to distinguish them. Validity is the notion we are really interested in, but we need the notion of tautology as a technical notion.

Proposition 4.15. *Every tautology is valid, but not vice versa.*

Proof. Suppose that θ is a tautology, and take an arbitrary interpretation $\mathfrak{A}$. We get a normal truth assignment by stipulating that, for any ϕ,

$$\begin{aligned} \mathfrak{F}(\phi) &= 1 \text{ if } \phi \text{ is true under } \mathfrak{A} \\ &= 0 \text{ otherwise} \end{aligned}$$

So $\mathfrak{F}(\theta) = 1$. Hence θ is true under $\mathfrak{A}$. Since $\mathfrak{A}$ was arbitrary, this shows that every tautological formula is valid. On the other hand, the tautological formula "$((\forall x)Fx \rightarrow Fc)$ is not tautological. □

A tautological sentence is a valid sentence whose validity is determined by the sentence's truth functional structure. If, instead, the validity of a sentence's truth functional structure. If, instead, the validity of a sentence depends upon the meaning of the quantifiers, the sentence won't be tautological.

We can test whether a sentence is tautological by the method of truth tables, examining each possible way to assign a truth value to the sentences' basic truth functional components. Alternatively, we can test the sentence by the search-for-counterexample method. For example, to show that "$(((\exists x)Fx \rightarrow (\forall x)Gx \vee (\neg Hc \rightarrow (\exists x)Fx))$" is tautological, we have the following:

$$\underline{(((\exists x)Fx \rightarrow (\forall x)Gx) \vee (\neg Hc \rightarrow (\exists x)Fx))}$$

$$\cancel{1 \quad\quad 0 \quad 0 \quad\quad\quad 0 \quad\quad\quad 0 \quad X}$$

Definition 4.16. A sentence ϕ is a *logical consequence* of a set of sentences γ iff ϕ is true under every interpretation under which all the members of γ are true. ϕ is a *tautological consequence* of a set of sentences γ iff ϕ is assigned the value 1 by every N.T.A. which assigns the value 1 to every member of γ.

The same reasoning which gave us the last proposition yields the following:

Proposition 4.17. *Every tautological consequence of a set of sentences is a logical consequence, but not vice versa.*

The following definitions and theorems are lifted directly from the sentential calculus:

Definition 4.18. A sentence is *contradictory* (or *inconsisent*) iff it is false under every interpretation. A sentence is *indeterminate* iff it is true under some interpretations and false under others. A sentence ϕ *implies* (or *entails*) sentence ψ iff ψ is true under every interpretation under which ϕ is true. ϕ and ψ are *logically equivalent* iff they are true under precisely the same interpretations. An argument is *valid* iff the conclusion is true under every interpretation under which the premises are true. A set of sentences is *consistent* iff there is some interpretation under which all its members are true.

Theorem 4.19. *A sentence is a valid iff its negation is contradictory.*

Theorem 4.20. *A sentence is contradictory iff its negation is valid.*

Theorem 4.21. *A sentence is indeterminate iff its negation is indeterminate.*

Theorem 4.22. *A conjuction is valid iff both its conjuncts are valid.*

Theorem 4.23. *If a conjunction is contradictory if (but not necessarily only if) either of its conjuncts is.*

Theorem 4.24. *A disjunction is valid if(but not only if) either disjunct is valid.*

Theorem 4.25. *A disjunction is contradictory iff both disjuncts are contradictory.*

Theorem 4.26. *A conditional is contradictory iff its antecedent is valid and its consequent is a contradiction.*

Theorem 4.27. *Two sentences ϕ and ψ are logically equivalent iff the biconditional $(\phi \leftrightarrow \psi)$ is valid.*

Theorem 4.28. $\neg(\phi \vee \psi)$ *is logically equivalent to* $(\neg\phi \vee \neg\psi)$.

Theorem 4.29. $\neg(\phi \vee \psi)$ *is logically equivalent to* $(\neg\phi \vee \neg\psi)$.

Theorem 4.30. ϕ *implies* ψ *iff the conditional* $(\phi \rightarrow \psi)$ *is valid.*

Theorem 4.31. *A contradiction implies every sentence.*

Theorem 4.32. *A valid sentence is implied by every sentence.*

Theorem 4.33. *Two sentences are logically equivalent iff each implies the other.*

Theorem 4.34. *An argument is valid iff the conjunction of the premises entails the conclusion.*

Theorem 4.35. *An argument is valid iff the conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion is valid.*

Theorem 4.36. ϕ *is a logical consequence of* $\{\gamma_1, \gamma_1, \ldots, \gamma_n\}$ *if and only if the argument with* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *as premises and with* ϕ *as condition is valid.*

Theorem 4.37. *A sentence is a logical consequence of the empty set iff it is valid.*

Theorem 4.38. *A sentence is valid iff it is a logical consequence of that set of sentences.*

Theorem 4.39. *Each member of a set of sentences is a logical consequence of that set of sentences.*

Theorem 4.40. *If every member of* Δ *is a logical consequence of* Γ *and* ϕ *is a logical consequence of* Δ*, then* ϕ *is a logical consequence of* Γ.

Theorem 4.41. *If* Δ *is a subset of* Γ *and* ϕ *is a logical consequence of* Δ*, then* ϕ *is a logical consequence of* Γ.

Theorem 4.42. *For any sentence* ψ *and set of sentences* Γ*,* ψ *is a logical consequence of* Γ *if and only if* Γ *and* $\Gamma \cup \{\psi\}$ *have precisely the same logical consequences.*

Theorem 4.43. *$(\phi \wedge \psi)$ is a logical consequence of Γ iff ϕ and ψ are both logical consequences of Γ.*

Theorem 4.44. *$(\phi \to \psi)$ is a logical consequence of Γ iff ψ is a logical consequences of $\Gamma \cup \{\phi\}$.*

Theorem 4.45. *$\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is inconsistent iff $(\gamma_1 \wedge (\gamma_2 \wedge \ldots \wedge \gamma_n) \ldots)$ is an inconsistent sentence.*

Theorem 4.46. *If Γ is an inconsistent set of sentences, then every sentence is a logical consequence of Γ.*

Theorem 4.47. *A set of sentences Γ is inconsistent iff $(P \wedge \neg P)$ is a logical consequence of Γ.*

Theorem 4.48. *A set of sentences Γ is inconsistent iff every sentence is a logical consequence of Γ.*

Theorem 4.49. *If Δ is inconsistent and $\Delta \subseteq \Gamma$, then Γ is inconsistent.*

Theorem 4.50. *ϕ is a logical consequence of Γ iff $\Gamma \cup \{\neg\phi\}$ is inconsistent.*

Theorem 4.51 (Substitution Principle)**.** *For any interpretation $\mathfrak{A}$, individual constant c, and formula ϕ, $\phi\frac{x}{c}$ is true under $\mathfrak{A}$ iff $\mathfrak{A}$(c) satisfies ϕ under $\mathfrak{A}$.*

Proof. I am going to write out this proof in excruciating detail, just so you will see what one of these proofs looks like when written out in utter detail. I promise not to do it again.

Let $\mathfrak{A}$ be an interpretation and c a constant, and let Σ be the set of formulas ϕ such that $\phi\frac{x}{c}$ is true under $\mathfrak{A}$ iff $\mathfrak{A}$(c) satisfies ϕ under $\mathfrak{A}$. Clearly, {formulas} $\subseteq \Sigma$. But also, since {formulas} is the smallest class of expressions which contains the atomic formulas and which is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification, if we can show that Σ is a class of expressions which contains the atomic formulas and which is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification and existential quantification, this will tell us that {formulas} $\subseteq \Sigma$. This will tell us that {formulas} $= \Sigma$, which is what we want.

Lemma 4.52 (Atomic formlas are in Σ)**.** *If ϕ is an atomic formula, then either it has the form Fx or it has the form Fd. If ϕ has the form Fx, then $\phi\frac{x}{c}$ is Fc. We have*

$\phi\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff Fc *is true under* $\mathfrak{A}$
iff $\mathfrak{A}(c) \in \mathfrak{A}(F)$
iff $\mathfrak{A}(c)$ *satisfies* Fx) *under* $\mathfrak{A}$.

If ϕ *has the form* Fd, *then* $\phi\frac{x}{c} = \phi$. *We have*

$\phi\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff ϕ *is true under* $\mathfrak{A}$
iff $\mathfrak{A}(c)$ *satisfies* ϕ *under* $\mathfrak{A}$ *[by the corollary to the principle of bivalence (Corollary 4.12, page 24)].*

Lemma 4.53 (Σ is closed under conjunction)**.** *Suppose that* ϕ *and* ψ *are both in* Σ. *Then* $(\phi' \wedge' \psi\frac{x}{c})$, *and we have:*

$(\phi \wedge \psi)\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff $(\phi\frac{x}{c} \wedge \psi\frac{x}{c})$ *is true under* $\mathfrak{A}$
iff both $\phi\frac{x}{c}$ *and* $\psi\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff $\mathfrak{A}(c)$ *satisfies* ϕ *under* $\mathfrak{A}$ *and* $\mathfrak{A}(c)$ *satisfies* ψ *under* $\mathfrak{A}$ *[because* ϕ *and* ψ *are both in* Σ*]*
iff $\mathfrak{A}(c)$ *satisfies* $(\phi \wedge \psi)$ *under* $\mathfrak{A}$.

So $(\phi \wedge \psi)$ *is in* Σ.

Lemma 4.54 (Σ is closed under disjunction)**.** *Suppose that* ϕ *and* ψ *are both in* Σ. *Then* $(\phi' \vee' \psi)\frac{x}{c}$ *is equal to* $(\phi\frac{x}{c} \vee \psi\frac{x}{c})$, *and we have:*

$(\phi \vee \psi)\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff $(\phi\frac{x}{c} \vee \psi\frac{x}{c})$ *is true under* $\mathfrak{A}$
iff either $\phi\frac{x}{c}$ *and* $\psi\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff either $\mathfrak{A}(c)$ *satisfies* ϕ *under* $\mathfrak{A}$ *or* $\mathfrak{A}(c)$ *satisfies* ψ *under* $\mathfrak{A}$ *[because* ϕ *and* ψ *are both in* Σ*]*
iff $\mathfrak{A}(c)$ *satisfies* $(\phi \vee \psi)$ *under* $\mathfrak{A}$.

So $(\phi \vee \psi)$ *is in* Σ.

Lemma 4.55 (Σ is closed under the formation of conditionals)**.** *Suppose that* ϕ *and* ψ *are both in* Σ. *Then* $(\phi' \to' \psi)\frac{x}{c}$ *is equal to* $(\phi\frac{x}{c} \to \psi\frac{x}{c})$, *and we have:*

$(\phi \to \psi)\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff $(\phi\frac{x}{c} \to \psi\frac{x}{c})$ *is true under* $\mathfrak{A}$
iff either $\phi\frac{x}{c}$ *is not true under* $\mathfrak{A}$ *or* $\psi\frac{x}{c}$ *is true under* $\mathfrak{A}$
iff either $\mathfrak{A}(c)$ *does not satisfy* ϕ *under* $\mathfrak{A}$ *or* $\mathfrak{A}(c)$ *does satisfies* ψ *under* $\mathfrak{A}$ *[because* ϕ *and* ψ *are both in* Σ*]*
iff $\mathfrak{A}(c)$ *satisfies* $(\phi \to \psi)$ *under* $\mathfrak{A}$.

So $(\phi \to \psi)$ *is in* Σ.

Lemma 4.56 (Σ is closed under the formation of biconditionals)**.** *Suppose that ϕ and ψ are both in Σ. Then $(\phi' \leftrightarrow' \psi)\frac{x}{c}$ is equal to $(\phi\frac{x}{c} \leftrightarrow \psi\frac{x}{c})$, and we have:*

> *$(\phi \leftrightarrow \psi)\frac{x}{c}$ is true under $\mathfrak{A}$*
> *iff $(\phi\frac{x}{c} \leftrightarrow \psi\frac{x}{c})$ is true under $\mathfrak{A}$*
> *iff $\phi\frac{x}{c}$ and $\psi\frac{x}{c}$ are either both true under $\mathfrak{A}$ or both false under $\mathfrak{A}$*
> *iff either $\mathfrak{A}(c)$ satisfies both ϕ and ψ under $\mathfrak{A}$ or $\mathfrak{A}(c)$ satisfies neither ϕ nor ψ under $\mathfrak{A}$ [because ϕ and ψ are both in Σ]*
> *iff $\mathfrak{A}(c)$ satisfies $(\phi \leftrightarrow \psi)$ under $\mathfrak{A}$.*

So $(\phi \leftrightarrow \psi)$ is in Σ.

Lemma 4.57 (Σ is closed under negation)**.** *Suppose that ϕ is in Σ. Then $(\neg\phi)\frac{x}{c}$ is equal to $\neg(\phi\frac{x}{c})$, and we have:*

> *$(\neg\phi)\frac{x}{c}$ is true under $\mathfrak{A}$*
> *iff $\neg(\phi\frac{x}{c})$ is true under $\mathfrak{A}$*
> *iff $\phi\frac{x}{c}$ is not true under $\mathfrak{A}$*
> *iff $\mathfrak{A}(c)$ does not satisfy ϕ under $\mathfrak{A}$ [because ϕ and ψ are both in Σ]*
> *iff $\mathfrak{A}(c)$ satisfies $\neg\phi$ under $\mathfrak{A}$.*

So $\neg\phi$ is in Σ.

Lemma 4.58 (Σ is closed under universal quantification)**.** *Suppose ϕ is in Σ. $((\forall x)\phi)\frac{x}{c}$ is equal to $(\forall x)\phi$, and we have:*

> *$((\forall x)\phi)\frac{x}{c}$ is true under $\mathfrak{A}$*
> *iff $(\forall x)\phi$ is true under $\mathfrak{A}$*
> *iff $\mathfrak{A}(c)$ satisfies $(\forall x)\phi$ under $\mathfrak{A}$ [by the corollary to the principle of bivalence (Corollary 4.12, page 24)].*

So $(\forall x)\phi$ is in Σ.

Lemma 4.59 (Σ is closed under existential quantification)**.** *Suppose ϕ is in Σ. $((\exists x)\phi)\frac{x}{c}$ is equal to $(\exists x)\phi$, and we have:*

> *$((\exists x)\phi)\frac{x}{c}$ is true under $\mathfrak{A}$*
> *iff $(\exists x)\phi$ is true under $\mathfrak{A}$*
> *iff $\mathfrak{A}(c)$ satisfies $(\exists x)\phi$ under $\mathfrak{A}$ [by the corollary to the principle of bivalence (Corollary 4.12, page 24)].*

So $(\exists x)\phi$ is in Σ. □

If our language has just three predicates, "F," "G," and "H," then any interpretation of the language divides the universe into 8 cells, numbered 1 through 8 in the figure (where some of the cells may be empty). If two members of the universe lie in the same cell, they satisfy all the same formulas. This observation is perfectly general:

Theorem 4.60 (Indiscernibility Principle)**.** *Given an interpretation* $\mathfrak{A}$*, let us say that two elements of* $|\mathfrak{A}|$ *are in the same cell if they are in the extensions in* $\mathfrak{A}$ *of all the same predicates. Any two members of* $|\mathfrak{A}|$ *that are in the same cell satisfy all the same formulas under* $\mathfrak{A}$*.*

Proof. Suppose that a and b are in the same cell. Let Σ be the set of formulas ϕ such that a satisfies ϕ under $\mathfrak{A}$ iff b satisfies ϕ under $\mathfrak{A}$. We want to see that Σ is equal to the set of all formulas. To show this, we need to show that Σ contains the atomic formulas and that it is closed under conjunction, disjunction, formation of conditionals, biconditionals, negation, universal quantification, and existential quantification.

Lemma 4.61 (Atomic formulas of the form Px are in Σ)**.** *Because a and b are in the same cell, we know that $a \in \mathfrak{A}(P)$ iff $b \in \mathfrak{A}(P)$. Thus a satisfies Px under $\mathfrak{A}$ iff b satisfies Px under $\mathfrak{A}$.*

Lemma 4.62 (Atomic sentences—formulas of the form Pc—are in Σ)**.** *If Pc is true, a and b both satisfy it. It it is false, neither does.*

Lemma 4.63 (Σ is closed under conjunction)**.** *Suppose that ϕ and ψ are both in Sigma. We have*

> *a satisfies $(\phi \wedge \psi)$ under $\mathfrak{A}$*
> *iff a satisfies both ϕ and ψ under $\mathfrak{A}$*
> *iff b satisfies both ϕ and ψ under $\mathfrak{A}$ [because ϕ and ψ are both in Σ]*
> *iff b satisfies $(\phi \wedge \psi)$ in Σ.*

So $(\phi \wedge \psi)$ is in Σ.

Lemma 4.64 (Σ is closed under disjunction, formation of conditional, and formation of biconditionals.)**.** *Similar.*

Lemma 4.65 (Σ is closed under universal quantification.)**.** *Suppose that ϕ is in Σ. We have*

> *a satisfies $(\forall x)\phi$ under $\mathfrak{A}$*
> *iff $(\forall x)\phi$ is true under $\mathfrak{A}$ [by the corollary to bivalence, (Corollary 4.12, page 24)]*
> *iff b satisfies $(\forall x)\phi$ under $\mathfrak{A}$ [by the corollary to bivalence again].*

So $(\forall x)\phi$ in in Σ.

Lemma 4.66 (Σ is closed under existential quantification)**.** *Similar.*

□

To see whether a sentence is true under an interpretation, you have to see what the universe of the interpretation is, and you have to see what values the interpretation assigns to the constants and predicates that appear within that sentence. That's all you have to look at. You don't have to look at the values the interpretation assigns to the constants and predicates that don't even occur within the sentence. The following theorem makes this observation precise:

Theorem 4.67 (Locality Principle)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two interpretations with the same universe of discourse that assign the same values to all the constants and predicates that occur in the formula ϕ. Then precisely the same individuals satisfy ϕ under $\mathfrak{A}$ and under $\mathfrak{B}$.*

Proof. Given interpretations $\mathfrak{A}$ and $\mathfrak{B}$ with the same universe of discourse, let $\Sigma = \{$formulas ϕ: if $\mathfrak{A}$ and $\mathfrak{B}$ assign the same values to all the constants and predicates that occur in ϕ, then the same individuals satisfy ϕ under $\mathfrak{A}$ and under $\mathfrak{B}\}$. We want to show that Σ is the set of all formulas. To show this, it will be enough to show that Σ contains the atomic formulas and that it is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification.

Lemma 4.68 (Σ contains the atomic formulas)**.** *Let ϕ be an atomic formula such that any constant or predicate that appears in ϕ is assigned the same value by $\mathfrak{A}$ and by $\mathfrak{B}$. Take $a \in |\mathfrak{A}|$. Either ϕ has the form Fx or else it has the form Fc.*

If ϕ has the form Fx, we have

a satisfies ϕ under $\mathfrak{A}$
iff $a \in \mathfrak{A}(F)$
iff $a \in \mathfrak{B}(F)$
iff a satisfies ϕ under $\mathfrak{B}$.

If ϕ has the form Fc, we have

a satisfies ϕ under $\mathfrak{A}$
iff $\mathfrak{A}(c) \in \mathfrak{A}(F)$
iff $\mathfrak{B}(c) \in \mathfrak{B}(F)$
iff a satisfies ϕ under $\mathfrak{B}$.

Lemma 4.69 (Σ is closed under conjunction)**.** *Suppose that ϕ and ψ are both in Σ, and take $a \in |\mathfrak{A}|$. Suppose that any constant or predicate that occurs in $(\phi \wedge \psi)$ is assigned the same value by $\mathfrak{A}$ and $\mathfrak{B}$. Then every constant or predicate that occurs in ϕ is assigned the same value by $\mathfrak{A}$ and by $\mathfrak{B}$, so that, since ϕ is in Σ, a satisfies ϕ under $\mathfrak{A}$ iff a satisfies ϕ under $\mathfrak{B}$. Similarly for ψ. Hence*

a satisfies $(\phi \wedge \psi)$ *under* $\mathfrak{A}$
iff a satisfies both ϕ *and* ψ *under* $\mathfrak{A}$
iff a satisfies both ϕ *and* ψ *under* $\mathfrak{B}$
iff a satisfies $(\phi \wedge \psi)$ *under* $\mathfrak{B}$*.*

Lemma 4.70 (Σ is closed under disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification)**.** *Similar.* □

The Indiscernibility Principle talks about members of the domain of a single interpretation. There is a generalization of it that applies across intepretations:

Theorem 4.71 (Generalized Indiscernibility Principle)**.** *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be interpretations of a MPC language* $\mathscr{L}$*. Suppose that precisely the same cells are nonempty in* $\mathfrak{A}$ *and in* $\mathfrak{B}$*. That is, for each element* a *of* $|\mathfrak{A}|$*, there is an element of* b *of* $|\mathfrak{B}|$ *such that, for any predicate* P *of* $\mathscr{L}$*,* $a \in \mathfrak{A}(P)$ *iff* $b \in \mathfrak{B}(P)$*, and likewise, for each element* b *of* $|\mathfrak{B}|$*, there is an element* a *of* $|\mathfrak{A}|$ *such that* a *and* b *are in the extensions of the same predicates in their respective models. Suppose further that, for each constant* c *of* $\mathscr{L}$*,* $\mathfrak{A}(c)$ *is in the same cell in* $\mathfrak{A}$ *that* $\mathfrak{B}(c)$ *is in* $\mathfrak{B}$*. Then the same sentences are true in* $\mathfrak{A}$ *and in* $\mathfrak{B}$

Proof. We prove something stronger, namely, that if a occupies the same cell in $\mathfrak{A}$ that b occupies in $\mathfrak{B}$, then a satisfies all the same formulas in $\mathfrak{A}$ that b satisfies in $\mathfrak{B}$. To show this, let Σ be the set of formulas ϕ such that, whenever an element a of $|\mathfrak{A}|$ occupies the same cell in $\mathfrak{A}$ that $b \in |\mathfrak{B}|$ occupies in $\mathfrak{B}$, a satisfies ϕ in $\mathfrak{A}$ iff it satisfies ϕ in $\mathfrak{B}$. We intend to show that every formula is in Σ, by showing that Σ contains the atomic sentences and is closed under the seven procedures for building larger formulas out of smaller ones.

Lemma 4.72 (Atomic formulas of the form Px are in Σ)**.** *What it means to say that* a *occupies the same cell in* $\mathfrak{A}$ *that* b *occupies in* $\mathfrak{B}$ *is that, for each predicate* P*,* $a \in \mathfrak{A}(P)$ *iff* $b \in \mathfrak{B}(P)$*. Consequently,* a *satisfies* Px *under* $\mathfrak{A}$ *iff* b *satisfies* Px *under* $\mathfrak{B}$*.*

Lemma 4.73 (Atomic formulas of the form Pc are in Σ)**.** *By hypothesis,* $\mathfrak{A}(C)$ *occupies the same cell in* $\mathfrak{A}$ *that* $\mathfrak{B}(C)$ *occupies in* $\mathfrak{B}$ *Consequently, we have:*

a satisfies Pc *in* $\mathfrak{A}$
iff Pc *is true in* $\mathfrak{A}$
iff $\mathfrak{A}(c) \in \mathfrak{A}(P)$
iff $\mathfrak{B}(c) \in \mathfrak{B}(P)$
iff Pc *is true in* $\mathfrak{B}$

iff b *satisfies* Pc *in* $\mathfrak{B}(P)$

Lemma 4.74 (Σ is closed under conjunction)**.** *Suppose that* ψ *and* θ *are in* Σ *and that* a *and* b *occupy the same cells in their respective interpretations. We have:*

> *a satisfies* $(\psi \wedge \theta)$ *in* $\mathfrak{A}$
> *iff* a *satisfies both* ψ *and* θ *in* $\mathfrak{A}$
> *iff* b *satisfies both* ψ *and* θ *in* $\mathfrak{B}$ *[because* ψ *and* θ *are in* Σ*]*
> *iff* b *satisfies* $(\psi \wedge \theta)$ *in* $\mathfrak{B}$

Lemma 4.75 (Σ is closed under disjunction, conditionals, biconditional, and negations)**.** *Similar.*

Lemma 4.76 (Σ is closed under existential quantification)**.** *Suppose that* ψ *is in* Σ*, and that* a *and* b *occupy the same cells in their respective models. Suppose that* a *satisfies* $(\exists x)\psi$ *in* $\mathfrak{A}$*. Then* $(\exists x)\psi$ *is true in* $\mathfrak{A}$*, which means that there is an element* c *of* $|\mathfrak{A}|$ *that satisfies* ψ *in* $\mathfrak{A}$*. By hypothesis, there is an element* d *of* $|\mathfrak{B}|$ *that occupies the same cell in* $\mathfrak{B}$ *that* c *occupies in* $\mathfrak{A}$*. Because* ψ *is in* Σ*, it follows that* d *satisfies* ψ *in* $\mathfrak{B}$*. Consequently,* $(\exists x)\psi$ *is true in* $\mathfrak{B}$*, and so* b *satisfies* $(\exists x)\psi$ *in* $\mathfrak{B}$*. The converse - if* b *satisfies* $(\exists x)\psi$ *in* $\mathfrak{B}$*,* a *satisfies* $(\exists x)\psi$ *in* $\mathfrak{A}$ *- is similar.*

Lemma 4.77 (Σ is closed under universal quantification)**.** □

If our language has n predicates, then an interpretation $\mathfrak{A}$ of the language partitions the universe of $\mathfrak{A}$ into 2^n cells, which we can number 1 through 2^n. (Some of the cells may be empty.) Form an interpretation $\mathfrak{B}$ as follows:

$$\begin{aligned}
|\mathfrak{B}| &= \{\text{numbers } k\text{: under } \mathfrak{A}\text{, the } k\text{th cell is empty}\}.\\
\mathfrak{B}(F) &= \{\text{numbers } k\text{: the } k\text{th cell is a nonempty part of } \mathfrak{A}(F)\},\\
&\quad\ \text{for F a predicate.}\\
\mathfrak{B}(c) &= \text{the number } k \text{ such that } \mathfrak{A}(c) \text{ is in the kth cell, for } c \text{ an}\\
&\quad\ \text{individual constant.}
\end{aligned}$$

It follows from the Generalized Indiscernibility Principle that the same sentences are true in $\mathfrak{A}$ and in $\mathfrak{B}$. Let us call $\mathfrak{B}$ the *canonical model* associated with $\mathfrak{A}$.

As an example, let $\mathscr{L}$ be the language whose predicates are “M”, “W”, and “F” and whose only individual constant is “t”’, and let $\mathfrak{A}$ be the interpretation of $\mathscr{L}$ given by:

$$\begin{array}{rcl} |\mathfrak{A}| & = & \{\text{animals}\} \\ \mathfrak{A} & = & \{\text{mammals}\} \\ \mathfrak{A} & = & \{\text{warm-blooded animals}\} \\ \mathfrak{A} & = & \{\text{animals that fly}\} \\ \mathfrak{A} & = & \text{Termin the dog} \end{array}$$

We assign numbers to the cells by specifying, for each number between 1 and 8, the combination of atomic and negated atomic formulas of the form Px that are satisfied by the occupants of that cell. Thus

Occupants of cell 1 satisfy "$(Mx \wedge (Wx \wedge Fx))$."
Occupants of cell 2 satisfy "$(Mx \wedge (Wx \wedge \neg Fx))$."
Occupants of cell 3 satisfy "$(Mx \wedge (\neg Wx \wedge Fx))$."
Occupants of cell 4 satisfy "$(Mx \wedge (\neg Wx \wedge \neg Fx))$."
Occupants of cell 5 satisfy "$(\neg Mx \wedge (Wx \wedge Fx))$."
Occupants of cell 6 satisfy "$(\neg Mx \wedge (Wx \wedge \neg Fx))$."
Occupants of cell 7 satisfy "$(\neg Mx \wedge (\neg Wx \wedge Fx))$."
Occupants of cell 8 satisfy "$(\neg Mx \wedge (\neg Wx \wedge \neg Fx))$."

There are animals that occupy each of the cells other than 3 and 4. Bats, for example, occupy cell 1, dogs cell 2, canaries cell 5, penguins cell 6, butterflies cell 7, and banana slugs cell 8. Tarmin is in cell 2.

Our canonical model $\mathfrak{B}$ associated with $\mathfrak{A}$ looks like this:

$$\begin{array}{rcl} |\mathfrak{B}| & = & \{1,2,5,6,7,8\} \\ \mathfrak{B}(\text{“}M'') & = & \{1,2\} \\ \mathfrak{B}(\text{“}W'') & = & \{1,2,5,6\} \\ \mathfrak{B}(\text{“}F'') & = & \{1,5,7\} \\ \mathfrak{B}(\text{“}t'') & = & 2 \end{array}$$

Theorem 4.78. *Given a language with finitely many predicates, a sentence is valid iff it is true in all the canonical models for that language. A sentence is a logical consequence of a set of sentences* Γ *iff the sentence is true in every canonical model in which each member of* Γ *is true.*

A sentence containing n predicates is valid iff it is true under every interpretation whose universe is contained in the set $\{1, 2, 3, \ldots, 2^n\}$.

How many canonical models are there? Given a language $\mathscr{L}$ with n predicates and k constants, the domain of a canonical model which will be a nonempty subset of $\{1, 2, 3, \ldots, 2^n\}$ there are 2^{2^n} such nonempty subsets. If the model has i nonempty cells, there will be i^k ways to apportion k constants into nonempty cells. Consequently, the total number of canonical models for $\mathscr{L}$ will be:

$$\sum_{i=1}^{2^n} \binom{i}{2^n} \cdot i^k$$

where $\binom{i}{2^n}$ is the binomial coefficient, which counts the number of ways to choose i elements from a 2^n element set. Thus there are a great many canonical models. Even so, there are only finitely many, and each of them is a finite structure, so we could, in principle, test a sentence for validity by examining all the canonical models to see if we can find one in which the given sentence is false. This gives us the following:

Corollary 4.79. *There is an algorithm—that is, a mechanical procedure—for testing whether a sentence is valid.*

The algorithm just desribed is not at all practical, for the canonical interpretations are far too numerous for it to be feasible to examine them all. The theoretical possibility of testing a sentence ϕ for validity by examining all the models of ϕ whose universe is contained within $\{1, 2, 3, \ldots, 2^n\}$ remains only that, a theoretical possibility.

In the next chapter, we are going to learn a more practical method for showing valid sentences valid.

If our language has n predicates and k constants, we can write down, for each interpretation of the language, a *canonical description* that completely describes the model. The canonical description is a long conjunction with two kinds of conjuncts. First, there are conjuncts that tell us, for each individual constant, which cell the individual named by the constant occupies. For our model above that had as its sole constant "t" denoting Tarmin, the conjunct would be "$(Mt \wedge (Wt \wedge \neg Ft))$." Second, for each of the cells not occupied saying whether the cell is empty. Thus for cell 1, the conjunct is "$(\exists x)(Mx \wedge (Wx \wedge Fx))$," whereas for cell 3 the conjunct is "$\neg(\exists x)(Mx \wedge (\neg Wx \wedge Fx))$." The canonical description of the model is the following long sentence:

$$\begin{aligned}
&(Mt \wedge (Wt \wedge \neg Ft) \wedge ((\exists x)(Mx \wedge (Wx \wedge Fx)) \\
&\wedge(\neg(\exists x)(Mx \wedge (\neg Wx \wedge Fx)) \wedge (\neg(\exists x)(Mx \wedge (\neg Fx \wedge \\
&\quad \neg Wx)) \\
&\wedge((\exists x)(\neg Mx \wedge (Wx \wedge Fx)) \wedge ((\exists x)(\neg Mx \wedge (Wx \wedge \neg Fx)) \\
&\wedge((\exists x)(\neg Mx \wedge (\neg Wx \wedge Fx)) \\
&\wedge(\exists x)(\neg Mx \wedge (\neg Wx \wedge \neg Fx))))))))).
\end{aligned}$$

We know from the Generalized Indiscernibility Principle that determining which cells the individuals named by the constants are in and which of the remaining cells are nonempty is enough to decide which sentences are true. It follows, for every sentence, a canonical description entails either the sentence or its negation. What we say is that the canonical descriptions are *complete*. Every complete, consistent sentence is logically equivalent to a canonical description.

For a given sentence ϕ (still talking about a language with n predicates and k constants), the *normal form* of ϕ is the disjunction of all the canonical descriptions of models in which ϕ is true. For any model $\mathfrak{A}$ of ϕ, the canonical description of $\mathfrak{A}$ will be one of the disjuncts of the normal form of ϕ, and so the normal form of ϕ will be true in $\mathfrak{A}$. Consequently, ϕ entails its normal form. Moreover, each of the disjuncts of the normal form of ϕ entails ϕ, and so their disjunction entails ϕ. It follows that a sentence is logically equivalent to its normal form. (If ϕ happens to be a contradiction, there will not be any disjuncts. In this case, we can choose, say "$(\exists x)(Mx \wedge \neg Mx)$" to be the normal form of ϕ.

Every sentence is logically equivalent to its normal form, and no two normal forms are logically equivalent, since they are true in different models. If there are m canonical models, there will be 2^m normal forms, one for every subset of the m models. It follows that it is possible to give a list of 2^m sentences such that no two sentences on the list are logically equivalent and such that every sentence of the language is logically equivalent to a sentence on the list.