Derived Rule for Substitution of Equivalents

Our system of rules in complete; if φ is a consequence of Γ, then there is a derivation of φ with a premiss set that is included in Γ. The system is also reasonably efficient, so long as we restrict ourselves to inferences that contain only the connectives "$\neg$" and "$\rightarrow$," but the clumsiness of rule (DC) makes inferences involving the other connectives awkward. Now we'll introduce a new derived rule that makes things easier.

> **Derived Rule for the Substitution of Equivalents (SE).** Suppose that φ has been derived from the premiss set Γ, that $(\chi \leftrightarrow \theta)$ has been derived with premiss set Δ, and that ψ has been obtained from φ either by replacing χ with θ or by replacing θ with χ. Then you may derive ψ with premiss set $\Gamma \cup \Delta$.

This is a "derived" rule because everything you can prove with the rule you can prove more laboriously without it. I'll postpone the proof of this for a while.

For rule SE to be useful, you need a large supply of biconditional theorems on hand. The basic strategy for proving biconditionals is clear. If we want to prove $(\varphi \leftrightarrow \psi)$, we first prove $(\varphi \rightarrow \psi)$ and then prove $(\psi \rightarrow \varphi)$. TH16 gives us $((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow (\varphi \leftrightarrow \psi)))$, so we derive $(\varphi \leftrightarrow \psi)$ by two applications of MP. In practice, the relevant instance of TH16 is long enough that writing it out is a real nuisance, enough so that we adopt a new derived rule:

> **Derived Rule for Biconditional Introduction (BI).** If you have derived both $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$, you may write $(\varphi \leftrightarrow \psi)$, taking as premiss set the union of the premiss sets of the two conditionals.

As the simplest possible example, we derive "$(P \leftrightarrow P)$":

1	1. P	PI
TH23	2. $(P \rightarrow P)$	CP, 1
TH24	3. $(P \leftrightarrow P)$	BI, 2

A similar derivation gives us "$(P \leftrightarrow \neg\neg P)$":

	1. $(\neg\neg P \rightarrow P)$	TH1
	2. $(P \rightarrow \neg\neg P)$	TH5
TH25	3. $(P \leftrightarrow \neg\neg P)$	BI, 1, 2

In the same way:

	1. $((P \vee Q) \rightarrow (\neg P \rightarrow Q)$	DC
	2. $((\neg P \rightarrow Q) \rightarrow (P \vee Q))$	DC
TH26	3. $((P \vee Q) \leftrightarrow (\neg P \rightarrow Q))$	BI, 1, 2

	1. $((P \wedge Q) \rightarrow \neg (P \rightarrow \neg Q))$	DC
	2. $(\neg (P \rightarrow \neg Q) \rightarrow (P \wedge Q))$	DC
TH27	3. $((P \wedge Q) \leftrightarrow \neg (P \rightarrow \neg Q))$	BI 1, 2

	1. $((P \leftrightarrow Q) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P)))$	DC
	2. $(((P \rightarrow Q) \wedge (Q\rightarrow P)) \rightarrow (P \leftrightarrow Q))$	DC
TH28	3. $((P \wedge Q) \leftrightarrow \neg (P \rightarrow \neg Q))$	BI, 1, 2

We'll next prove the Import-Export Law, "$((P \rightarrow (Q \rightarrow R) \leftrightarrow ((P \wedge Q) \rightarrow R)$." The left-to-right direction (TH29) is called "importation" — "P" is imported into the antecedent — and the right-to-left direction (TH30) is called "exportation:

1	1. $(P \rightarrow (Q \rightarrow R))$	PI
2	2. $(P \wedge Q)$	PI
	3. $((P \wedge Q) \rightarrow P)$	TH13
2	4. P	MP, 2, 3
1,2	5. $(Q \rightarrow R)$	MP, 1, 4
	6. $((P \wedge Q) \rightarrow Q)$	TH14
2	7. Q	MP, 2, 6
1,2	8. R	MP, 5, 7
1	9. $((P \wedge Q) \rightarrow R)$	CP, 2, 8
TH29	10. $((P \rightarrow (Q \rightarrow R)) \rightarrow ((P \wedge Q) \rightarrow R))$	CP, 2, 9
11	11. $((P \wedge Q) \rightarrow R)$	PI
12	12. P	PI
13	13. Q	PI
	14. $(P \rightarrow (Q \rightarrow (P \wedge Q)))$	TH15
12	15. $(Q \rightarrow (P \wedge Q))$	MP, 12, 14
12,13	16. $(P \wedge Q)$	MP, 13, 15
11,12,13	17. R	MP. 11, 16
11,12	18. $(Q \rightarrow R)$	CP,13, 17
11	19. $(P \rightarrow (Q \rightarrow R))$	CP, 12, 18
TH30	20. $(((P \wedge Q) \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R)))$	CP, 11, 19
TH31	21. $((P \rightarrow (Q \rightarrow R)) \leftrightarrow ((P \wedge Q) \rightarrow R))$	BI 10, 20

As a first application of SE, let's prove another of de Morgan's laws:

	1. $(\neg (\neg P \wedge \neg Q) \leftrightarrow (\neg \neg P \vee \neg \neg Q)$	TH22
	2. $(P \leftrightarrow \neg \neg P)$	TH24
	3. $(Q \leftrightarrow \neg \neg Q)$	TH24
	4. $(\neg (\neg P \wedge \neg Q) \leftrightarrow (P \vee \neg \neg Q))$	SE 1, 2
	5. $(\neg (\neg P \wedge \neg Q) \leftrightarrow (P \vee Q))$	SE 3, 4
	6. $(\neg (P \vee Q) \leftrightarrow \neg (P \vee Q))$	TH23
	7. $(\neg (P \vee Q) \leftrightarrow \neg \neg (\neg P \wedge \neg Q))$	SE 5, 6
	8. $((\neg P \wedge \neg Q) \leftrightarrow \neg \neg (\neg P \wedge \neg Q))$	TH24
TH32	9. $(\neg (P \vee Q) \leftrightarrow (\neg P \wedge \neg Q))$	SE 7, 8

Now let's prove the commutative, associative, and idempotent laws for "$\vee$":

	1. $((\neg P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg\neg P)$	TH19
	2. $((P \lor Q) \leftrightarrow (\neg P \rightarrow Q))$	TH26
	3. $((P \lor Q) \leftrightarrow (\neg Q \rightarrow \neg\neg P))$	SE 1, 2
	4, $((Q \lor \neg\neg P) \leftrightarrow (\neg Q \rightarrow \neg\neg P))$	TH26
	5. $((P \lor Q) \leftrightarrow (Q \lor \neg\neg P))$	SE 2, 4
	6. $(P \leftrightarrow \neg\neg P)$	TH25
TH33	7. $((P \lor Q) \leftrightarrow (Q \lor P))$	SE 5, 6

	1. $((\neg P \rightarrow (\neg Q \rightarrow R)) \leftrightarrow ((\neg P \land \neg Q) \rightarrow R))$	TH31
	2. $((Q \lor R) \leftrightarrow (\neg Q \rightarrow R))$	TH26
	3. $((\neg P \rightarrow (Q \lor R)) \leftrightarrow ((\neg P \land \neg Q) \rightarrow R))$	SE 1, 2
	4. $((P \lor (Q \lor R)) \leftrightarrow (\neg P \rightarrow (Q \lor R))$	TH26
	5. $((P \lor (Q \lor R)) \leftrightarrow ((\neg P \land \neg Q) \rightarrow R))$	SE 3, 4
	6. $(\neg (P \lor Q) \leftrightarrow (\neg P \land \neg Q))$	TH32
	7. $((P \lor (Q \lor R)) \leftrightarrow (\neg (P \lor Q) \rightarrow R))$	SE 5, 6
	8. $(((P \lor Q) \lor R) \leftrightarrow (\neg (P \lor Q) \rightarrow R)$	TH25
TH34	9. $((P \lor (Q \lor R)) \leftrightarrow ((P \lor Q) \lor R)$	SE 7, 8

	1. $((\neg P \rightarrow P) \rightarrow P)$	TH7
	2. $((P \lor P) \leftrightarrow (\neg P \rightarrow P))$	TH26
TH35	3. $((P \lor P) \rightarrow P)$	SE 1, 2
	4. $(P \rightarrow (P \lor P))$	TH10
TH36	5. $((P \lor P) \leftrightarrow P)$	BI 2, 3

We now want to derive the commutative, associative, and idempotent laws for "∧." A bit of gimmickry will save us from having to do any hard work. Using de Morgan's laws, we can derive the principles we want from the corresponding principles for "∨."

	1. $((\neg P \lor \neg Q) \leftrightarrow (\neg Q \lor \neg P))\_$	TH33
	2. $(\neg (P \land Q) \leftrightarrow (\neg P \lor \neg Q))$	TH22
	3. $(\neg (P \land Q) \leftrightarrow (\neg Q \lor \neg P))$	SE 1, 2
	4. $(\neg (Q \land P) \leftrightarrow (\neg Q \lor \neg P))$	TH22
	5. $(\neg(P \land Q) \leftrightarrow \neg(Q \land P))$	SE 3, 4
	6. $(\neg\neg (P \land Q) \leftrightarrow \neg\neg (P \land Q))$	TH24
	7. $(\neg\neg (P \land Q) \leftrightarrow \neg\neg (Q \land P))$	SE 5, 6
	8. $((P \land Q) \leftrightarrow \neg\neg (P \land Q))$	TH25
	9. $((P \land Q) \leftrightarrow \neg\neg (Q \land P))$	SE 5, TH25
	10. $((Q \land P) \leftrightarrow \neg\neg (Q \land P))$	TH25
TH37	11. $((P \land Q) \leftrightarrow (Q \land P))$	SE 9, 10

	1. $((\neg P \lor (\neg Q \lor \neg R)) \leftrightarrow ((\neg P \lor \neg Q) \lor \neg R))$	TH34
	2. $(\neg (Q \land R) \leftrightarrow (\neg Q \lor \neg R))$	TH22
	3. $((\neg P \lor \neg (Q \land R)) \leftrightarrow ((\neg P \lor \neg Q) \lor \neg R))$	SE 1, 2
	4. $(\neg (P \land (Q \land R)) \leftrightarrow (\neg P \lor \neg (Q \land R)))$	TH22
	5. $(\neg (P \land (Q \land R)) \leftrightarrow ((\neg P \lor \neg Q) \lor \neg R))$	SE 3, 4
	6. $(\neg (P \land Q) \leftrightarrow (\neg P \lor \neg Q))$	TH22
	7. $(\neg (P \land (Q \land R)) \leftrightarrow (\neg (P \land Q) \lor \neg R))$	SE 5, 6
	8. $(\neg ((P \land Q) \land R) \leftrightarrow (\neg (P \land Q) \lor \neg R))$	TH22
	9. $(\neg (P \land (Q \land R)) \leftrightarrow \neg ((P \land Q) \land R))$	SE 7, 8
	10. $(\neg \neg (P \land (Q \land R)) \leftrightarrow \neg \neg (P \land (Q \land R)))$	TH24
	11. $(\neg \neg (P \land (Q \land R)) \leftrightarrow \neg \neg ((P \land Q) \land R))$	SE 9, 10
	12. $((P \land (Q \land R)) \leftrightarrow \neg \neg (P \land (Q \land R)))$	TH25
	13. $((P \land (Q \land R)) \leftrightarrow \neg \neg ((P \land Q) \land R))$	SE 11, 12
	14. $(((P \land Q) \land R) \leftrightarrow \neg \neg ((P \land Q) \land R))$	TH25
TH38	15. $((P \land (Q \land R)) \leftrightarrow ((P \land Q) \land R))$	SE 13, 14

	1. $((\neg P \lor \neg P) \leftrightarrow \neg P)$	TH36
	2. $(\neg (P \land P) \leftrightarrow (\neg P \lor \neg P))$	TH22
	3. $(\neg (P \land P) \leftrightarrow \neg P)$	SE 1, 2
	4. $(\neg \neg (P \land P) \leftrightarrow \neg \neg (P \land P))$	TH24
	5. $(\neg \neg (P \land P) \leftrightarrow \neg \neg P)$	SE 4, 3
	6. $((P \land P) \leftrightarrow \neg \neg (P \land P))$	TH25
	7. $((P \land P) \leftrightarrow \neg \neg P)$	SE 5, 6
	8. $(P \leftrightarrow \neg \neg P)$	TH25
TH39	9. $((P \land P) \leftrightarrow P)$	SE 7, 8

Now we prove one of the distributive laws, "$((P \land (Q \lor R)) \leftrightarrow ((P \land Q) \lor (P \land R)))$":

1	1. $(P \land (Q \lor R))$	PI
	2. $((P \land (Q \lor R)) \rightarrow P)$	TH13
1	3. P	MP, 1, 2
	4. $((P \land (Q \lor R)) \rightarrow (Q \lor R))$	TH14
1	5. $(Q \lor R)$	MP 1, 4
6	6. Q	PI
	7. $(P \rightarrow (Q \rightarrow (P \land Q)))$	TH15
1	8. $(Q \rightarrow (P \land Q))$	MP 3, 7
1,6	9. $(P \land Q)$	MP 6, 8
	10. $((P \land Q) \rightarrow ((P \land Q) \lor (P \land R)))$	TH10
1,6	11. $((P \land Q) \lor (P \land R))$	MP 9 ,10
1	12. $(Q \rightarrow ((P \land Q) \lor (P \land R)))$	CP 6, 11
13	13. R	PI
	14 $(P \rightarrow (R \rightarrow (P \land R))$	TH15
1	15. $(R \rightarrow (P \land R))$	MP 3, 14

1,13	16 $(P \wedge R)$	MP 13, 15
	17. $((P \wedge R) \rightarrow ((P \wedge Q) \vee (P \wedge R)))$	TH11
1,13	18. $((P \wedge Q) \vee (P \wedge R))$	MP 16, 17
1	19. $(R \rightarrow ((P \wedge Q) \vee (P \wedge R)))$	CP 13, 18
	20. $((Q \rightarrow ((P \wedge Q) \vee (P \wedge R))) \rightarrow ((R \rightarrow ((P \wedge Q) \vee (P \wedge R))) \rightarrow ((Q \vee R) \rightarrow ((P \wedge Q) \vee (P \wedge R)))))$	TH12
1	21. $((R \rightarrow ((P \wedge Q) \vee (P \wedge R))) \rightarrow ((Q \vee R) \rightarrow ((P \wedge Q) \vee (P \wedge R))))$	MP 12, 20
1	22. $((Q \vee R) \rightarrow ((P \wedge Q) \vee (P \wedge R)))$	MP 19, 21
1	23. $((P \wedge Q) \vee (P \wedge R))$	MP 5, 22
TH40	24. $((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R)))$	CP 1, 23
25	25. $((P \wedge Q) \vee (P \wedge R))$	PI
26	26. $(P \wedge Q)$	PI
	27. $((P \wedge Q) \rightarrow P)$	TH13
26	28. P	MP 26, 27
	29. $((P \wedge Q) \rightarrow Q)$	TH14
26	30. Q	MP 26, 29
	31. $(Q \rightarrow (Q \vee R))$	TH10
26	32. $(Q \vee R)$	MP 30, 31
	33. $(P \rightarrow ((Q \vee R) \rightarrow (P \wedge (Q \vee R))))$	TH15
26	34. $((Q \vee R) \rightarrow (P \wedge (Q \vee R)))$	MP 28, 33
26	35. $(P \wedge (Q \vee R))$	MP 32, 34
	36 $((P \wedge Q) \rightarrow (P \wedge (Q \vee R)))$	CP 26, 35
37	37. $(P \wedge R)$	PI
	38. $((P \wedge R) \rightarrow P)$	TH13
37	39. P	MP 37, 38
	40. $((P \wedge R) \rightarrow R)$	TH14
37	41. R	MP 37, 40
	42. $(R \rightarrow (Q \vee R))$	TH11
37	43. $(Q \vee R)$	MP 41, 42
37	44. $((Q \vee R) \rightarrow (P \wedge (Q \vee R)))$	MP 33, 39
37	45. $(P \wedge (Q \vee R))$	MP 43, 44
	46. $((P \wedge R) \rightarrow (P \wedge (Q \vee R)))$	CP 37, 45
	47. $(((P \wedge Q) \rightarrow (P \wedge (Q \vee R))) \rightarrow (((P \wedge R) \rightarrow (P \wedge (Q \vee R))) \rightarrow (((P \wedge Q) \vee (P \wedge R)) \rightarrow (P \wedge (Q \vee R)))))$	TH12
	48. $(((P \wedge R) \rightarrow (P \wedge (Q \vee R))) \rightarrow (((P \wedge Q) \vee (P \wedge R)) \rightarrow (P \wedge (Q \vee R))))$	MP 36, 47
TH41	49. $(((P \wedge Q) \vee (P \wedge R)) \rightarrow (P \wedge (Q \vee R)))$	MP 46, 48
TH42	50. $((P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee (P \wedge R)))$	BI 24, 48

Now for the other distributive law, "$((P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R)))$":

1. $((\neg P \wedge (\neg Q \vee \neg R)) \leftrightarrow ((\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)))$	TH42
2. $(\neg (Q \wedge R) \leftrightarrow (\neg Q \vee \neg R))$	TH22
3. $((\neg P \wedge \neg (Q \wedge R)) \leftrightarrow ((\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)))$	SE 1, 2
4. $(\neg (P \vee (Q \wedge R)) \leftrightarrow (\neg P \wedge \neg (Q \wedge R)))$	TH32

	5. $(\neg (P \vee (Q \wedge R)) \leftrightarrow ((\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)))$	SE 3, 4
	6. $(\neg (P \vee Q) \leftrightarrow (\neg P \wedge \neg Q))$	TH32
	7. $(\neg (P \vee (Q \wedge R)) \leftrightarrow (\neg (P \vee Q) \vee (\neg P \wedge \neg R)))$	SE 5, 6
	8. $(\neg (P \vee R) \leftrightarrow (\neg P \wedge \neg R))$	TH32
	9. $(\neg(P \vee (Q \wedge R)) \leftrightarrow (\neg(P \vee Q) \vee \neg (P \vee R)))$	SE 7, 8
	10. $(\neg((P \vee Q) \wedge (P \vee R)) \leftrightarrow (\neg(P \vee Q) \vee \neg (P \vee R)))$	TH22
	11. $(\neg (P \vee (Q \wedge R)) \leftrightarrow \neg ((P \vee Q) \wedge (P \vee R)))$	SE 9 ,10
	12. $(\neg \neg (P \vee (Q \wedge R)) \leftrightarrow \neg \neg (P \vee (Q \wedge R)))$	TH24
	13. $(\neg \neg (P \vee (Q \wedge R)) \leftrightarrow \neg ((P \vee Q) \wedge (P \vee R)))$	SE 11, 12
	14. $((P \vee (Q \wedge R)) \leftrightarrow \neg \neg (P \vee (Q \wedge R))$	TH25
	15. $((P \vee (Q \wedge R)) \leftrightarrow \neg \neg ((P \vee Q) \wedge (P \vee R)))$	SE, 13, 14
	16. $(((P \vee Q) \wedge (P \vee R)) \leftrightarrow \neg \neg ((P \vee Q) \wedge (P \vee R)))$	TH25
TH43	17. $((P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R)))$	SE 15, 16

It's now time to think about the status of SE as a derived rule. The most straightforward way of showing that rule SE doesn't give us anything genuinely new is by a detailed examination of the ways the formula φ might have been constructed. showing that, no matter how the formula φ was built up, the formula ψ. The required examination would be entirely unproblematic, but lengthy.

Because we have the soundness and completeness theorem in hand, and because the proof of the theorem didn't employ SE, a shorter argument is available. Under the hypotheses of the rule, φ is derivable from Γ and either $(\chi \leftrightarrow \theta)$ or $(\theta \leftrightarrow \chi)$ is derivable from Δ. It follows by soundness that φ is a logical consequence of Γ and that either $(\chi \leftrightarrow \theta)$ or $(\theta \leftrightarrow \chi)$ is a logical consequence of Δ. SC Theorem 10 informs us that, under these circumstances, ψ is a logical consequence of $\Gamma \cup \Delta$, and hence, according to the completeness theorem, ψ is derivable (without using SE) from $\Gamma \cup \Delta$.

The more direct proof is more informative than the argument that appeals to the completeness theorem, since it shows us how to convert a proof that employs SE into a proof that does not. The argument from completeness tells us that a proof without SE exists, but it doesn't tell us how to find it. So the short cut through completeness loses information. On the other hand, life is short.

Basic Rules of Deduction

PI You may write down any sentence you like if you take the sentence as its own premiss set.

CP If you have derived ψ with premiss set Γ, you may write $(\phi \to \psi)$ with premiss set $\Gamma \sim \{\varphi\}$.

MP If you have derived φ with premiss set Γ and $(\varphi \to \psi)$ with premiss set Δ, you may write ψ with premiss set $\Gamma \cup \Delta$.

MT If you have derived ψ with premiss set Γ and $(\neg \varphi \to \neg \psi)$ with premiss set Δ, you may write φ with premiss set $\Gamma \cup \Delta$.

DC You may write an instance of any of the following six schemata with the empty premiss set:
$((\varphi \vee \psi) \to (\neg \varphi \to \psi))$
$((\neg \varphi \to \psi) \to (\varphi \vee \psi))$
$((\varphi \wedge \psi) \to \neg(\varphi \to \neg \psi))$
$(\neg(\varphi \to \neg \psi) \to (\varphi \wedge \psi))$
$((\varphi \leftrightarrow \psi) \to ((\varphi \to \psi) \wedge (\psi \to \varphi)))$
$(((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\varphi \leftrightarrow \psi))$

Derived Rules

TH If you have already proved φ from the empty set, you may, at any time in any derivation, write down any substitution instance of φ again, with the empty premiss set.

SE Suppose that φ has been derived from the premiss set Γ, that $(\chi \leftrightarrow \theta)$ has been derived with premiss set Δ, and that ψ has been obtained from φ either by replacing χ with θ or by replacing θ with χ. Then you may derive ψ with premiss set $\Gamma \cup \Delta$.

BI If you have derived both $(\varphi \to \psi)$ and $(\psi \to \varphi)$, you may write $(\varphi \leftrightarrow \psi)$, taking as premiss set the union of the premiss sets of the two conditionals.

SC Theorems We Have Proved Thus Far

TH1	$(\neg\neg P \rightarrow P)$	Double negation elimination
TH2	$(Q \rightarrow (P \rightarrow Q))$	
TH3	$((P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)))$	Principle of the syllogism
TH4	$(((P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$	
TH5	$(P \rightarrow \neg\neg P)$	Double negation introduction
TH6	$(\neg P \rightarrow (P \rightarrow Q))$	Law of Duns Scotus
TH7	$((\neg P \rightarrow P) \rightarrow P)$	Law of Clavius
TH8	$(((P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R)))$	
TH9	$((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$	
TH10	$(P \rightarrow (P \vee Q))$	A disjunction introduction principle
TH11	$(Q \rightarrow (P \vee Q))$	A disjunction introduction principle
TH12	$((P \rightarrow R) \rightarrow ((Q \rightarrow R) \rightarrow ((P \vee Q) \rightarrow R)))$	Principle of disjunctive syllogism
TH13	$((P \wedge Q) \rightarrow P)$	A conjunction elimination principle
TH14	$((P \wedge Q) \rightarrow Q)$	A conjunction elimination principle
TH15	$(P \rightarrow (Q \rightarrow (P \wedge Q))$	Conjunction introduction principle
TH16	$((P \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow (P \leftrightarrow Q)))$	
TH17	$((P \leftrightarrow Q) \rightarrow (P \rightarrow Q))$	
TH18	$((P \leftrightarrow Q) \rightarrow (Q \rightarrow P))$	
TH19	$((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P))$	
TH20	$((\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q))$	
TH21	$((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$	Principle of contraposition
TH22	$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$	One of de Morgan's laws
TH23	$(P \rightarrow P)$	
TH24	$(P \leftrightarrow P)$	
TH25	$(P \leftrightarrow \neg\neg P)$	
TH26	$((P \vee Q) \leftrightarrow (\neg P \rightarrow Q))$	
TH27	$((P \wedge Q) \leftrightarrow \neg (P \rightarrow \neg Q))$	
TH28	$((P \wedge Q) \leftrightarrow \neg (P \rightarrow \neg Q))$	
TH29	$((P \rightarrow (Q \rightarrow R)) \rightarrow ((P \wedge Q) \rightarrow R))$	
TH30	$(((P \wedge Q) \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R)))$	
TH31	$((P \rightarrow (Q \rightarrow R)) \leftrightarrow ((P \wedge Q) \rightarrow R))$	
TH32	$(\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q))$	One of de Morgan's laws
TH33	$((P \vee Q) \leftrightarrow (Q \vee P))$	Commutative law for "$\vee$"
TH34	$((P \vee (Q \vee R)) \leftrightarrow ((P \vee Q) \vee R)$	Associative law for "$\vee$"
TH35	$((P \vee P) \rightarrow P)$	
TH36	$((P \vee P) \leftrightarrow P)$	Idempotence for "$\vee$"
TH37	$((P \wedge Q) \leftrightarrow (Q \wedge P))$	Commutative law for "$\wedge$"
TH38	$((P \wedge (Q \wedge R)) \leftrightarrow ((P \wedge Q) \wedge R))$	Associative law for "$\wedge$"
TH39	$((P \wedge P) \leftrightarrow P)$	Idempotence for "$\wedge$"
TH40	$((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R)))$	
TH41	$(((P \wedge Q) \vee (P \wedge R)) \rightarrow (P \wedge (Q \vee R)))$	
TH42	$((P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee (P \wedge R)))$	Distributive law
TH43	$((P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R)))$	Another distributive law