The Vitali Sets

1 Non-Measurable Sets

- There are subsets of $\mathbb{R}$ that are **non-measurable**:

  They cannot be assigned a measure by any extension of λ, without giving up on Non-Negativity, Countable Additivity, or Uniformity.

2 The Axiom of Choice

Proving that there are non-measurable sets requires:

Axiom of Choice Every set of non-empty, non-overlapping sets has a choice set.

(A **choice set** for set A is a set that contains exactly one member from each member of A.)

3 Defining the Vitali Sets

3.1 A sketch of the construction

- Define an (uncountable) partition $\mathcal{U}$ of $[0, 1)$.
- Use the Axiom of Choice to pick a representative from each cell of $\mathcal{U}$.
- Use these representatives to define a (countable) partition $\mathcal{C}$ of $[0, 1)$.
- A Vitali Set is a cell of $\mathcal{C}$.

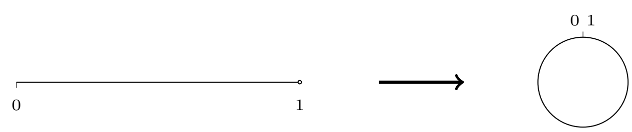
3.2 Defining $\mathcal{U}$

$a, b \in [0, 1)$ are in the same cell if and only if $a - b \in \mathbb{Q}$.

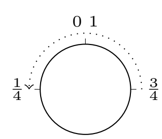
3.3 Defining $\mathcal{C}$

- $\mathcal{C}$ has a cell C_q for each rational number $q \in \mathbb{Q}^{[0,1)}$.
- C_0 is the set of representatives of cells of $\mathcal{U}$.
- C_q is the set of numbers $x \in [0, 1)$ which are at a "distance" of q from the representative of their cell in $\mathcal{U}$.

Here "distance" is measured by bending $[0, 1)$ into a circle:



and traveling counter-clockwise. For instance, $\frac{1}{4}$ is at "distance" $\frac{1}{2}$ from $\frac{3}{4}$:



4 A Vitali Set Cannot Be Measured

4.1 Assumptions

Countable Additivity

$$\lambda\left(\bigcup\{A_1, A_2, A_3, \ldots\}\right) = \lambda(A_1) + \lambda(A_2) + \lambda(A_3) + \ldots$$

whenever $A_1, A_2, \ldots$ is a countable family of disjoint sets for each of which λ is defined.

Non-Negativity $\lambda(A)$ is either a non-negative real number or the infinite value ∞, for any set A in the domain of λ.

Uniformity $\mu(A^c) = \mu(A)$, whenever $\mu(A)$ is well-defined and A^c is the result of adding $c \in \mathbb{R}$ to each member of A.

4.2 The Proof

- Suppose otherwise: $\lambda(C_q)$ is well-defined for some $q \in \mathbb{Q}^{[0,1)}$.
- By Uniformity, $\lambda(C'_q) = \lambda(C_q)$ for any $q' \in \mathbb{Q}^{[0,1)}$.
- By Non-Negativity, $\lambda(C_q)$ is either 0, or a positive real number, or ∞.
- By Countable Additivity, it can't be any of these:
  - Suppose $\lambda(C_q) = 0$. By Countable Additivity:

$$\begin{aligned} \lambda([0,1)) &= \lambda(C_q) + \lambda(C_{q'}) + \ldots \\ &= \underbrace{0 + 0 + 0 + \ldots}_{\text{once for each integer}} \\ &= 0 \end{aligned}$$

  - Suppose $\lambda(C_q) = r > 0$. By Countable Additivity:

$$\begin{aligned} \lambda([0,1)) &= \lambda(C_q) + \lambda(C_{q'}) + \ldots \\ &= \underbrace{r + r + r + \ldots}_{\text{once for each integer}} \\ &= \infty \end{aligned}$$

Moral: There is no way of assigning a measure to a Vitali set without giving up on Uniformity, Non-Negativity or Countable Additivity.

5 Revising Our Assumptions?

- Giving up on **Uniformity** means *changing the subject*: the whole point of our enterprise is to find a way of extending the notion of Lebesgue Measure without giving up on uniformity.
- **Non-Negativity** and **Countable Additivity** are not actually needed to prove the existence of non-measurable sets.
- Some mathematical theories would be seriously weakened by giving up on the **Axiom of Choice**.