24.118: Paradox and Infinity, Spring 2019

Problem Sets 7 and 8: Non-Measurable Sets

How these problems will be graded:

- In Part I there is no need to justify your answers. Assessment will be based on whether your answers are correct.
- In Part II you must justify your answers. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on the basis of how your answer is justified. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- *No answer may consist of more than 150 words.* Longer answers will not be given credit. (Showing your work in a calculation, a proof, or a computer program does not count towards the word limit.)
- You may consult published literature and the web. You must, however, credit all sources. Failure to do so constitutes plagiarism and can have serious consequences. For advice about when and how to credit sources see: https://integrity.mit.edu (You do not need to credit course materials.)

Preliminaries

The line segment $[a, b]$ is the set of real numbers x such that $a \leq x \leq b$. The Borel Sets are the members of the smallest set $\mathscr{B}$ such that: (i) every line segment is in $\mathscr{B}$, (ii) if a set A is in $\mathscr{B}$, then so is $\mathbb{R} - A$, and (iii) if a countable family of sets is in $\mathscr{B}$, then so is its union.

The Lebesgue Measure, λ, is the unique function on the Borel Sets that satisfies these three conditions:

Length on Segments $\lambda([a, b]) = b - a$ for every $a, b \in \mathbb{R}$.

Countable Additivity

$$\lambda\left(\bigcup\{A_1, A_2, A_3, \ldots\}\right) = \lambda(A_1) + \lambda(A_2) + \lambda(A_3) + \ldots$$

whenever $A_1, A_2, \ldots$ is a countable family of disjoint sets for each of which λ is defined.

Non-Negativity $\lambda(A)$ is either a non-negative real number or the infinite value ∞, for any set A in the domain of λ.

Part I

1. Of each of the following sets of real numbers, determine whether it is a Borel Set. If it is a Borel Set, specify its Lebesgue Measure. (You may assume that every Borel Set has a Lebesgue Measure.) (2 points each)

   (a) $\left[\frac{1}{4}, \frac{1}{3}\right]$
   (b) $\{\frac{1}{3}\}$
   (c) $\left[\frac{1}{4}, \frac{1}{3}\right)$
   (d) $\left[\frac{1}{4}, \frac{5}{6}\right] - \left[\frac{1}{3}, \frac{1}{2}\right]$
   (e) $\left[\frac{1}{4}, \frac{5}{6}\right] - \{\frac{1}{3}, \frac{1}{2}\}$
   (f) $\{0, 1, 2, \ldots\}$
   (g) $\left[\frac{4}{5}, \frac{5}{6}\right] \cup \{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$
   (h) $\left[0, \frac{1}{2}\right] - \{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$
   (i) a Vitali set
   (j) the complement of a Vitali set

   (As usual, $[a, b] = \{x : a \leq x \leq b\}$ and $[a, b) = \{x : a \leq x < b\}$.)

Part II

2. *Choosing Socks, Choosing Shoes*

   Recall that a **choice set** for set A is a set containing exactly one element from each member of A. And recall the Axiom of Choice:

   Axiom of Choice Any set of non-empty, non-overlapping sets has a choice set.

   On a first reading, the Axiom of Choice is likely to sound trivial. This exercise is aimed at helping you understand why it is not. (It is a variant of an explanation given long ago by British philosopher Bertrand Russell.)

   Here are two standard set-theoretic axioms:

   Union If a set A exists, then so does its union, $\bigcup A$.
   (Recall that $\bigcup A$ is the set $\{x : x$ is a member of some element of $A\}$, i.e. the set of members of members of A.)

   Separation Let $\phi(x)$ be any formula of the form "x is such and such" (for instance, "x is a natural number"). Then if set A exists, the following set also exists: $\{x : x \in A$ and $\phi(x)\}$ (i.e. the set of objects that are members of A and satisfy condition $\phi(x)$).

Here is an example of how Separation might be used. Let $\phi(x)$ be the formula "x is female". Separation entails that if the set $O = \{x : x \text{ is an octopus}\}$ exists, so does

$$\{x : x \in O \text{ and } x \text{ is female}\}$$

which is the set of female octopuses.

(a) Let S be an infinite set, each member of which is a set of two shoes: a right shoe and a left shoe. Assume that no two elements of S have any shoes in common. Now suppose you'd like to have a choice set for S. The Axiom of Choice guarantees that a choice set exists, but it doesn't give you much information about what it looks like.

Let's see if we can do better than that. It follows from Union that $\bigcup S$ exists. Is there an application of Separation to $\bigcup S$ that delivers a choice set for S? If so, sketch a proof. If not, explain informally why not. (5 points)

(b) Let S be an infinite set, each member of which is a set of two socks. We will assume that all socks are alike, and, in particular, that there is no such thing as a "right" sock or a "left" sock. (We will also assume, unrealistically, that socks are not located in space and therefore that socks cannot be distinguished by their spatial locations.) Assume that no two elements of S have any socks in common. Now suppose you'd like to have a choice set for S. The Axiom of Choice guarantees that a choice set exists, but it doesn't give you much information about what it looks like.

Let's see if we can do better than that. It follows from Union that $\bigcup S$ exists. Is there an application of Separation to $\bigcup S$ that delivers a choice set for S? If so, sketch a proof. If not, explain informally why not. (5 points)

3. *Back to Bacon*

This problem is about Bacon's Puzzle, which is discussed in Section 3.4 of the textbook. (Before reading further, you might consider having a look at the text, to refresh your memory of the puzzle.)

Towards the end of the discussion in the book, I write:

> What is the probability that an individual who follows the strategy will answer correctly? I don't know the answer to this question but I suspect that when one follows the strategy one's probability of success is best thought of ill-defined. (Section 3.4.9)

Throughout this problem, I will explain why I harbor such suspicions.

Let S be the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. We partition S into orbits, as follows: for any $f, g \in S$, f is in the same orbit as g if and only if there are at most finitely many numbers k such that $f(k) \neq g(k)$.

(a) Let $f_0(n) = 0$ for each $n \in \mathbb{N}$, and let O_0 be f_0's orbit. Describe a bijection μ_0 from $\mathbb{N}$ to O_0. (5 points)

(b) Given an arbitrary orbit O and a function $f^* \in O$, describe a bijection μ from $\mathbb{N}$ to O. (5 points)

The lesson of problem (3b) is that any representative from a given orbit can be used to define a well-ordering of that orbit.

Let us now consider the problem of how one might go about choosing a representative from each orbit. Ask yourself: is the set of orbits analogous to the set of pairs of shoes of problem (2a), or is it analogous to the set of pairs of socks of problem (2b)? In other words: is there a formula $\phi(x)$ such that an application of Separation based on $\phi(x)$ could be used to specify a set that contains exactly one representative for each orbit?

As it turns out, the answer to this question is "no". It is impossible to set forth an explicit rule that singles out exactly one representative from each orbit: the only way to show that a set of representatives exists is to use the Axiom of Choice.

(c) *Extra Credit:* Show that one cannot prove that a set with exactly one representative from each orbit exists without using the Axiom of Choice. You may avail yourself of the following important result, due to Robert Solovey: one cannot prove that a non-measurable set exists without using Axiom of Choice. (5 points)

Back to Bacon's Puzzle. The question we wish to consider is: what is the probability that an individual who follows the strategy will answer correctly? To fix ideas, let the individual in question be P_0 and assume that she has been given a blue hat. Let the function $f_@$ represent the actual distribution of hats and let $O_@$ be $f_@$'s orbit. Then our question can be reformulated as follows: what is the probability that orbit $O_@$ was assigned a representative ρ such that $\rho(0) = 1$?

In fact, there is a natural way of answering this question, *relative* to a well-ordering of $O_@$. Let μ be a bijection from $\mathbb{N}$ to $O_@$. Then μ can be used to characterize the following probability function:

$$p(Z) \quad =_{df} \quad \lim_{n\to\infty} \frac{|Z \cap \{\mu(0), \mu(1), \ldots, \mu(n)\}|}{|\{\mu(0), \mu(1), \ldots \mu(n)\}|}$$

(Here Z is a subset of $O_@$. If you'd like a refresher on this type of probability function, see Section 6.4.1.2 of the textbook.)

Let X be the proposition that $O_@$ was assigned a representative ρ such that $\rho(0) = 1$. (Formally: $X = \{f \in O_@ : f(0) = 1\}$.) In the next couple of questions I'll ask you to calculate the value of $p(X)$ relative to different orderings.

(d) Suppose that $O_@$ is orbit O_0 from problem (3a) and that μ is the bijection μ_0 you gave in your answer to (3a). What is the value of $p(X)$? (5 points)

(e) For a given integer $k \geq 2$, define a bijection μ from $\mathbb{N}$ to O_0 such that $p(X) = \frac{1}{k}$. (10 points)

As problem (3e) suggests, you can get $p(X)$ to have *any value you want*, by picking a sufficiently devious μ. So the probability function $p(\ldots)$ can only be assumed to assign a sensible probability to proposition X if it is defined using a sensible choice of μ.

When it comes to particular orbits, you may well think that there are choices of μ that stand out as particularly natural. Perhaps you think that when it comes to the specific orbit O_0 of problem (3a), the choice of μ_0 that you supplied in your answer is a particularly natural way of ordering O_0. (Maybe it even delivers the comforting result that $p(X) = 0.5$.)

But what about the general case? Is there a *general recipe* that can be used to specify a natural ordering of each of our uncountably many orbits. Unfortunately, the answer is "no":

(f) Show that it is impossible to explicitly characterize a relation $<$ such that each orbit O is well-ordered by $<$. You may make use of problem (3c). (10 points)

In the absence of a recipe for specifying a natural ordering for each orbit, I have no idea how one might go about characterizing sensible probability functions over the members of our orbits. *That*'s why I suspect that the probability of success in Bacon's Puzzle, given that one follows the strategy, is not, in general, well-defined.

4. *The Square of Evil*[1]

Say that a **countable ordinal** is an ordinal with countably many members, and let $\aleph_1$ be the set of all countable ordinals. $\aleph_1$ is itself an ordinal. From this it follows that $\aleph_1$ must have uncountably many members. (For suppose otherwise, then $\aleph_1$ is a countable ordinal, and therefore a member of itself. But no ordinal is a member of itself.)

Think of the **Continuum Hypothesis** as the claim that $\aleph_1$ has the same cardinality as [0,1], and therefore that there is a bijection f from [0,1] to $\aleph_1$. Assume that the Continuum Hypothesis is true, and define the following ordering $<^e$ of [0,1]:

$$\text{for any } a, b \in [0,1],\ a <^e b \text{ if and only if } f(a) \in f(b)$$

Since the ordinals in $\aleph_1$ are well-ordered by $\in$, it is an immediate consequence of this definition that $<^e$ is a well-ordering of [0,1].

[1] The construction is due to the Polish mathematician Wacław Sierpiński. I learned about it in Frank Arntzenius, Adam Elga and John Hawthorne's "Bayesianism, Infinite Decisions, and Binding".

(a) Show that $<^e$ has the following additional property: for each $x \in [0,1]$, there are at most countably many $y \in [0,1]$ such that $y <^e x$. (5 points)

We will now use $<^e$ to color the unit square [0,1]×[0,1], using the following criterion:

> For each point $\langle x, y \rangle \in [0,1] \times [0,1]$, color $\langle x, y \rangle$ white if $x <^e y$, and black otherwise.

I will refer to the colored square as the **Square of Evil**. Now let $\langle x_0, y_0 \rangle$ be a particular point on the Square of Evil:

(b) How many white points are there in the row $\{\langle z, y_0 \rangle : z \in [0,1]\}$? (5 points)

(c) How many white points are there in the column $\{\langle x_0, z \rangle : z \in [0,1]\}$? (5 points)

Suppose that a point is selected at random from the Square of Evil. (It is selected by twice applying the Standard Coin Toss Procedure of Section 7.1.4.1 of the reading materials, once to pick an x coordinate, and once to pick a y coordinate.)

(d) Someone tells you which row the selected point is in, and asks you to bet on whether the selected point is black or white. How should you bet? (5 points)

(e) Someone tells you which column the selected point is in, and asks you to bet on whether the selected point is black or white. How should you bet? (5 points)

It will rain or snow, but you don't know which. If it rains, you should wear outfit A rather than outfit B. If it snows, you should also wear outfit A rather than outfit B. So you should wear outfit A!

Here is a generalization of that seemingly attractive idea:

Dominance Let Π be a set of possible states of the world over which you have no control. You know that exactly one member of Π obtains, but you don't know which. Suppose you have two options, A and B. Suppose, moreover, that for each $\pi \in \Pi$ you should choose A over B on the assumption that π obtains. Then you should choose A over B.

(f) Use the Square of Evil to show that Dominance is false. (10 points)