24.118: Paradox and Infinity, Spring 2019
Problem Set 1: Infinite Cardinalities

How these problems will be graded:

- In Part I there is no need to justify your answers. Assessment will be based on whether your answers are correct.
- In Part II you must justify your answers. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on the basis of how your answer is justified. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- *No answer may consist of more than 150 words.* Longer answers will not be given credit. (Showing your work in a calculation, a proof, or a computer program does not count towards the word limit.)
- You may consult published literature and the web. You must, however, credit all sources. Failure to do so constitutes plagiarism and can have serious consequences. For advice about when and how to credit sources see: https://integrity.mit.edu/

Part I

1. A **function** $f : A \longrightarrow B$ is an assignment of each member of A to some member of B. A function $f : A \longrightarrow B$ is an **injection** if $f(x) \neq f(y)$ whenever $x \neq y$, for $a, b \in A$. A function $f : A \longrightarrow B$ is a **surjection** if there is no member of B to which f fails to assign some member of A.

   For each of the following functions $f : A \longrightarrow B$ determine whether f is injective and whether it is surjective.

   (a) (2 points)

$$\begin{aligned} f(x) &= x+1 \\ A &= \mathbb{N} \\ B &= \mathbb{N} \end{aligned}$$

   (b) (2 points)

$$\begin{aligned} f(x) &= x+1 \\ A &= \mathbb{N} \\ B &= \mathbb{N} - \{0\} \end{aligned}$$

(c) (2 points)

$$\begin{aligned} f(x) &= x^2 \\ A &= \mathbb{Z} \\ B &= \mathbb{Z} \end{aligned}$$

(d) (2 points)

$$\begin{aligned} f(x) &= x+1 \\ A &= \mathbb{R} \\ B &= \mathbb{R} \end{aligned}$$

Notation: $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, $\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2, \dots\}$ is the set of integers, $\mathbb{R}$ is the set of real numbers, and $X - Y$ is the set of entities that are members of X but not of Y.

2. A **bijection** from set A to set B is a function from A to B which is both injective and surjective. Between which of the following pairs of infinite sets is it possible to construct a bijection?

   (a) The set of negative integers and the set of non-negative integers. (2 points.)

   (b) The set of prime numbers and the set of real numbers. (2 points.)

   (c) The set of prime numbers and the set of real numbers between 0 and 1. (2 points.)

   (d) The rational numbers and the set of rational numbers between 0 and 1. (2 points.)

3. The following principles give conflicting answers to cardinality questions:

   The Proper Subset Principle Suppose A is a proper subset of B. Then A and B are *not* of the same size: B has more members than A.

   The Bijection Principle Set A has the same size as set B if and only if there is a *bijection* from A to B.

   For each of the questions below determine which of the following answers is correct: "yes", "no", or "not determined".

   (a) Are $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$ of the same size, according to the Proper Subset Principle? Are they of the same size according to the Bijection Principle? (3 points.)

   (b) Are $\{0, 1, 2\}$ and $\{1, 2, 3, 4\}$ of the same size, according to the Proper Subset Principle? Are they of the same size according to the Bijection Principle? (3 points.)

   (c) Are the set of prime numbers and the set of natural numbers of the same size, according to the Proper Subset Principle? Are they of the same size according to the Bijection Principle? (3 points.)

Reminder: Although you'll need to think about the Proper Subset Principle for the purposes of this question, it won't be relevant for the rest of the problem set. In this class we follow Cantor—and current mathematical practice—in rejecting the Proper Subset Principle and assessing cardinality questions on the basis of the Bijection Principle.

4. Every room in Hilbert's Hotel is occupied. New guests show up.
   (a) There is one new guest for each rational number. Can all of them be accommodated without asking anyone to share a room? (3 points.)
   (b) There is one new guest for each real number. Can all of them be accommodated without asking anyone to share a room? (3 points.)

5. Describe a set that contains no integers but has the same cardinality as the set of integers. (4 points.)

Part II

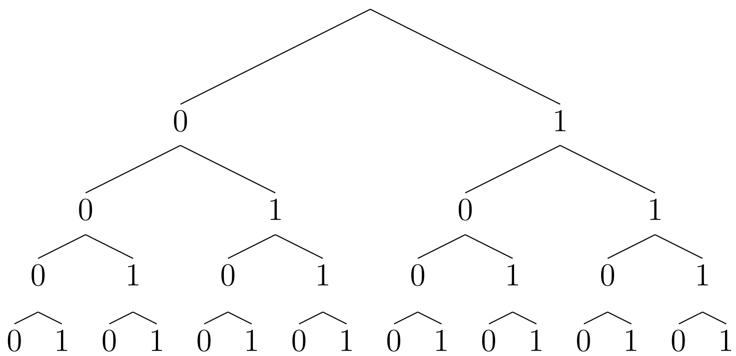
6. Show that there is a bijection between the set of integers {. . . -2, -1, 0, 1, 2, . . . } and the set of squares of integers $\{0, 1, 4, 9, 16, \dots\}$. (5 points)

7. Show that there is a bijection from $\mathbb{N}$ to $\{\langle n, m\rangle : n, m \in \mathbb{N}\}$, which is the set of pairs of natural numbers. (10 points)

8. Show that there cannot be a bijection from the set of natural numbers to the set of functions f such that $f : \mathbb{N} \longrightarrow \{-1, \pi, e\}$. (10 points)

9. The unit cube C is the set of triples $\langle r, p, q\rangle$, for $r, p, q \in [0, 1]$. A point $\langle r, p, q\rangle$ in C is said to be "rational" if all three of r, p, and q are rational.
   (a) Is there a bijection from the set of points in C to the set of real numbers? (10 points; don't forget to justify your answer)
   (b) Is there a bijection from the set of rational points in C to the set of rational numbers? (10 points; don't forget to justify your answer)

10. Consider the following infinite tree:



$$\vdots$$

(When fully spelled out, the tree contains one row for each natural number. The zero-th row contains one node, the first row contains two nodes, the second row contains four nodes, and, in general, the nth row contains 2^n nodes.)

(a) Is there a bijection between the *nodes* of this tree and the natural numbers? (10 points)

(b) Is there a bijection between the *paths* of this tree and the natural numbers? A path is an infinite sequence of nodes which starts at the top of the tree and contains a node at every row, with each node connected to its successor by an edge. (Paths can be represented as infinite sequences of zeroes and ones.) Don't forget to justify your answers! (10 points)