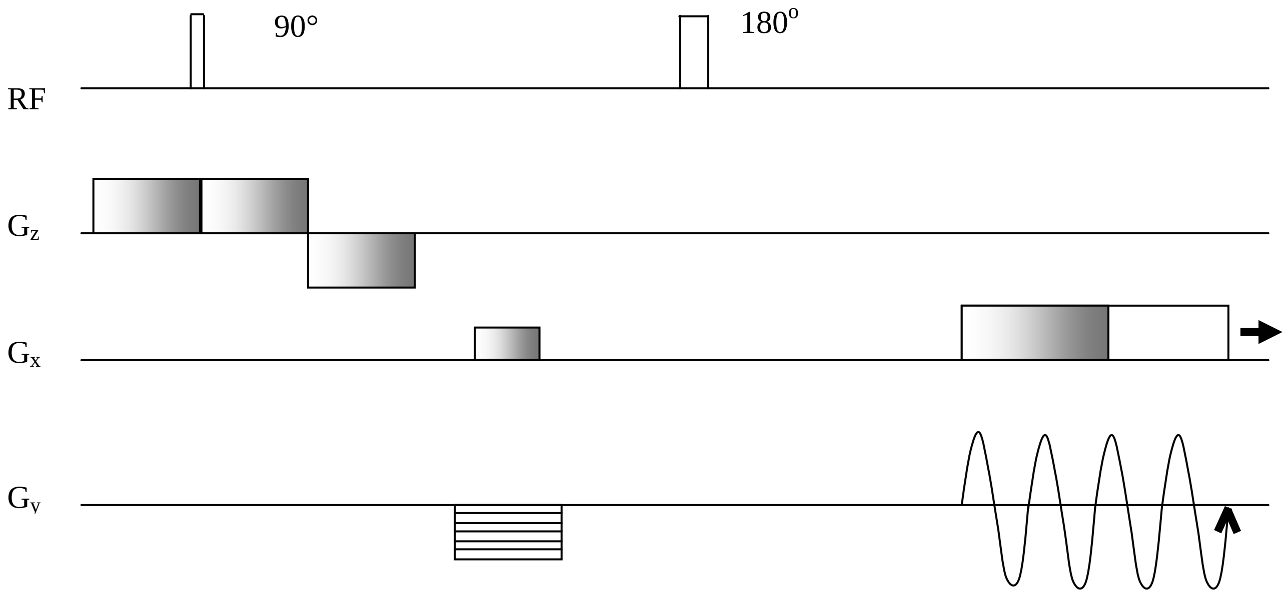
HST.584 / 22.561 Problem Set #4 Solutions

Figure & Fourier Transform proofs courtesy Mark Khachaturian

Marking Scheme: Question 1 – 4 points, Question 2 – 3 points, Question 3 – 3 points

1-a) The question alludes to a spin echo sequence. One possible implementation is:



If we examine the given k-space trajectory closely, we see that the sampling pattern repeats after every 4 samples (i.e. we return to the same value of k_y every 4^{th} sample). To generate 128 points in the frequency encode direction, we therefore must collect 4 x 128 = 512 points in each spin echo.

b) Each acquisition simultaneously acquires 4 different k_y lines in k-space, so we would need 128/4 = 32 repetitions of this basic pattern to generate our 128 phase encode steps. A standard 2DFT imaging sequence would require 128 steps so we have accelerated our sequence by a factor of 4. What we have done in general here is decreased the number of phase encode steps (phase encode steps are costly in terms of imaging time) by increasing the number of samples in the frequency encoding direction.

2) You are asked to prove any three of the Fourier transform theorems that Larry covered in class. Note that $F\{\ \}$ is used here as the Fourier transform operator.

Linearity: $F\{\alpha g + \beta h\} = \alpha F\{g\} + \beta F\{h\}$

$$
\begin{aligned}
F\{\alpha g+\beta h\} &= \iint [\alpha g(x,y)+\beta h(x,y)] e^{-2\pi i(k_x x+k_y y)} dxdy \\
&= \alpha \iint g(x,y) e^{-2\pi i(k_x x+k_y y)} dxdy + \beta \iint h(x,y) e^{-2\pi i(k_x x+k_y y)} dxdy \\
&= \alpha F(g)+\beta F(h)
\end{aligned}
$$

<u>Similarity:</u> $F\{g(ax,by)\} = \frac{1}{|a||b|} G\left(\frac{k_x}{a},\frac{k_y}{b}\right)$

$$
F\{g(ax,by)\} = \iint g(ax,by) e^{-2\pi i(k_x x+k_y y)} dxdy
$$

Substituting

$$
u = |a|x
$$

$$
v = |b|y
$$

gives

$$
\begin{aligned}
&= \frac{1}{|a||b|} \iint g(u,v) e^{-2\pi i\left(\frac{k_x u}{a}+\frac{k_y v}{b}\right)} dudv \\
&= \frac{1}{|a||b|} G\left(\frac{k_x}{a},\frac{k_y}{b}\right)
\end{aligned}
$$

<u>Shift Theorem:</u> $F\{g(x-a,y-b)\} = G(k_x,k_y) e^{-2\pi i(k_x a+k_y b)}$

$$
F\{g(ax,by)\} = \iint g(x-a,y-b) e^{-2\pi i(k_x x+k_y y)} dxdy
$$

Substituting

$$
u = x-a
$$

$$
v = y-b
$$

gives

$$
\begin{aligned}
&= \frac{1}{ab} \iint g(u,v) e^{-2\pi i(k_x u+k_y v)} e^{-2\pi i(k_x a+k_y b)} dudv \\
&= G(k_x,k_y) e^{-2\pi i(k_x a+k_y b)}
\end{aligned}
$$

<u>Parseval’s Theorem</u>: $\iint |g(x,y)|^2 dxdy = \iint |G(k_x,k_y)|^2 dk_x dk_y$

$$\iint |g(x,y)|^2\,dxdy = \iint g(x,y)g^*(x,y)dxdy$$
$$= \iint g(x,y)\left\{\iint G^*(k_x,k_y)e^{-2\pi i(k_x x+k_y y)}dk_x dk_y\right\}dxdy$$
$$= \iint G^*(k_x,k_y)\left\{\iint g(x,y)e^{-2\pi i(k_x x+k_y y)}dxdy\right\}dk_x dk_y$$
$$= \iint |G(k_x,k_y)|^2\,dk_x dk_y$$

Convolution Theorem: $F_{x,y}\left\{\iint g(a,b)h(x-a,y-b)dadb\right\} = G(k_x,k_y)H(k_x,k_y)$

$$F\left\{\iint g(a,b)h(x-a,y-b)dadb\right\} = \iint\left\{\iint g(a,b)h(x-a,y-b)dadb\right\}e^{-2\pi i(k_x x+k_y y)}dxdy$$
$$= \iint g(a,b)\left\{\iint h(x-a,y-b)e^{-2\pi i(k_x x+k_y y)}dxdy\right\}dadb$$

Using the shift theorem
$$= \iint g(a,b)H(k_x,k_y)e^{-2\pi i(k_x a+k_y b)}dadb$$
$$= G(k_x,k_y)H(k_x,k_y)$$
$$\Rightarrow F\{g(x,y)h(x,y)\} = G(k_x,k_y)H(k_x,k_y)$$

Fourier Integral Theorem: $FF^{-1}\{g(x,y)\} = F^{-1}F\{g(x,y)\} = g(x,y)$

$$FF^{-1}\{g(x,y)\} = \iint\left\{\iint g(x,y)e^{2\pi i(k_x x+k_y y)}dk_x dk_y\right\}e^{-2\pi i(k_x x+k_y y)}dxdy$$
$$= \iint\left\{\iint g(x,y)e^{-2\pi i(k_x x+k_y y)}dxdy\right\}e^{2\pi i(k_x x+k_y y)}dk_x dk_y$$
$$= F^{-1}F\{g(x,y)\}$$
$$= \iint G(k_x,k_y)e^{2\pi i(k_x x+k_y y)}dk_x dk_y = g(x,y)$$

3-a) In general, possible sources of inhomogeneities include main field inhomogeneities, susceptibility-induced variations and chemical shift effects to name a few. T_2* decay is a consequence of inhomogeneity (enhanced dephasing of spins within a given voxel) – this decay can both darken and blur an image. In the frequency encode direction, inhomogeneities can cause shifts in the image domain. In the phase encode direction, our image can acquire phase (although we can eliminate this by taking the magnitude of our image). We see this by writing our signal equation:

$$s(t) = \iint m(x,y)\exp(-i2\pi[k_x(t)x + k_y(t)y])\exp(-i\omega_E(x,y)t)dxdy$$

where ω_E represents the spatially dependent frequency effects due to any inhomogeneity. If we take m(x,y) to be a point object, we can track these effects more readily. Also, recall that in 2DFT we apply gradient G_y for some fixed interval t_y for phase encoding and G_x for a continuous time t during frequency encoding. Therefore:

$$s(t) = \exp(-i\gamma G_x x_0 t)\exp(-i\gamma G_y y_0 t_y)\exp(-i\gamma E(x_0, y_0)t)\exp(-i\phi_0)$$

where (x_0,y_0) is the location of our delta function object, $E(x_0, y_0) = \omega_E / \gamma$, and ϕ_0 is any extra phase accrued during the phase encode interval due to the inhomogeneity (note that it is not a function of time).

$$s(t) = \exp(-i\gamma[G_x x_0 + E(x_0, y_0)]t)\exp(-i\gamma G_y y_0 t_y)\exp(-i\phi_0)$$

$$= \exp(-i\gamma G_x[x_0 + E(x_0, y_0)/G_x]t)\exp(-i\gamma G_y y_0 t_y)\exp(-i\phi_0)$$

So, our object positioned at (x_0,y_0) will instead be mapped to $(x_0 + E(x_0,y_0)/G_x,y_0)$ with some extra phase factor added on to the image.

b) For the chemical shift, $E(x_0,y_0) = \omega_{CS} = \sigma B_0$. Therefore, our image will appear at $(x_0 + \sigma B_0 / G_x, y_0)$; in other words, for this type of imaging, the chemical shift will affect only the x-position and not the y-position. From the expression, we can see that we could reduced the chemical shift effect by increasing our gradient amplitude.