Problem Set #4

1. Pedestrian Light, revisited – 3 pts

Little's Law: L = λ.W and $L_q = \lambda.W_q$, with $\lambda = \lambda_L + \lambda_R$, $W = \frac{1}{u} + W_q$ and $L = L_q + \frac{\lambda}{u}$.

Rule A: $W_A = \frac{T}{2} \Rightarrow L_A = (\lambda_L + \lambda_R)\frac{T}{2}$

Rule B: $W_B = \frac{N_0 - 1}{2(\lambda_L + \lambda_R)} \Rightarrow L_B = \frac{N_0 - 1}{2}$

Rule C: $W_C = \frac{T_0}{2}\left(1 + \frac{1}{1 + (\lambda_L + \lambda_R).T_0}\right) \Rightarrow L_C = \frac{(\lambda_L + \lambda_R).T_0}{2}\left(1 + \frac{1}{1 + (\lambda_L + \lambda_R).T_0}\right)$

2. Squaring – 4 pts

a) **2 pts**

$$F_Y(y) = P[Y \le y] = P[X^2 \le y] = P\left[-\sqrt{y} \le X \le \sqrt{y}\right]$$

Let $F_Y(y)$ be the CDF of $f_Y(y)$:

$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x).dx = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)$$

Then, $f_Y(y) = \frac{d}{dy}\left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right] = \frac{1}{2.\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right]$

b) **2 pts**

$f_X(X) = \frac{1}{3}, \forall X \in [-1;2]$

Therefore, $f_Y(y) = \frac{1}{2.\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right] = \begin{cases} \frac{1}{3.\sqrt{y}}, \forall y \in [0;1] \\ \frac{1}{6.\sqrt{y}}, \forall y \in [1;4] \end{cases}$

***3. M/M/k Queue* – 6.5 pts**

a) **2 pts**

$$P(n) = \begin{cases} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} P_0, \forall n \in [0;k-1] \\ \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{k^{n-k}.k!} P_0, \forall n \in [k;\infty[\end{cases}$$

Since $\sum_{n=0}^{\infty} P_n = 1 \Leftrightarrow P_0 \left[\sum_{n=0}^{k-1} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=k}^{\infty} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{k^{n-k}.k!} \right] = 1$, we have: $P_0 = \dfrac{1}{\sum_{n=0}^{k-1} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=k}^{\infty} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{k^{n-k}.k!}}$

This becomes:

$$P_0 = \frac{1}{\sum_{n=0}^{k-1} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=k}^{\infty} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{k^{n-k}.k!}} = \frac{1}{\sum_{n=0}^{k-1} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} + \left(\dfrac{\lambda}{\mu}\right)^k \dfrac{1}{k!} \times \sum_{m=0}^{\infty} \left(\dfrac{\lambda}{\mu.k}\right)^m} = \frac{1}{\sum_{n=0}^{k-1} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} + \left(\dfrac{\lambda}{\mu}\right)^k \dfrac{1}{k!} \times \dfrac{1}{1 - \dfrac{\lambda}{\mu.k}}}$$

$$P_0 = \frac{1}{\sum_{n=0}^{k-1} \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n}{n!} + \dfrac{\lambda^k}{\mu^{k-1}.(\mu.k - \lambda)(k-1)!}}$$

We can then deduce the exact value for each P_n.

b) **1.5 pt**

$$L_q = \sum_{n=k}^{\infty} (n-k).P_n = P_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \times \sum_{m=0}^{\infty} m \left(\frac{\lambda}{k.\mu}\right)^m = P_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \times \frac{\dfrac{\lambda}{k.\mu}}{\left(1 - \dfrac{\lambda}{k.\mu}\right)^2} = \left(\frac{\lambda}{\mu}\right)^{k+1} \frac{P_0}{k.k! \left(1 - \dfrac{\lambda}{k.\mu}\right)^2}$$

Since $W = W_q + \dfrac{1}{\mu} = \dfrac{L_q}{\lambda} + \dfrac{1}{\mu}$,

$$W = \frac{\lambda^k.P_0}{\mu^{k+1}.k.k! \left(1 - \dfrac{\lambda}{k.\mu}\right)^2} + \frac{1}{\mu}$$

c) **1.5 pts**

A simulation shows that, when fixing $W_q = 1$ and $1/\mu = 1$ min, the progression of the maximum number of calls is faster than the progression of the number of servers required. In addition, the value (1-ρ) tends to 0.

d) **1.5 pts**
In such a case, the service quality is extremely sensitive to volume fluctuation. A mere 7% increase in calls and the call center can no longer service customers fast enough: the expected waiting time becomes infinite.

On February 27th, trading volumes exceeded the trading capacity of the stock market, creating an enormous queue of transactions. The value of stocks lagged far behind, creating confusion among traders.

4. ***M/M/k Queue with discouraged voters*** **– 6.5 pts**

a) **2 pts**

Since $\sum_{i=k}^{n}(i-k) = \frac{(n-k)(n-k+1)}{2}$

We have: $P(n) = \begin{cases} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} P_0, \forall n \in [0;k-1] \\ \frac{\left(\frac{\lambda}{\mu}\right)^n}{k^{n-k}.k!} P_0 \times e^{-\frac{(n-k)(n-k+1)}{20}}, \forall n \in [k;\infty[\end{cases}$

$$\sum_{n=0}^{\infty} P_n = 1 \Leftrightarrow P_0 \left[\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=k}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^n}{k^{n-k}.k!} \times e^{-\frac{(n-k)(n-k+1)}{20}} \right] = 1$$

This becomes: $P_0 = \frac{1}{\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=k}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^n}{k^{n-k}.k!} \times e^{-\frac{(n-k)(n-k+1)}{20}}} = \frac{1}{\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \times \sum_{m=0}^{\infty} \left(\frac{\lambda}{\mu.k}\right)^m \times e^{-\frac{m(m+1)}{20}}}$

b) **1.5 pts**

$$\lambda_{Av} = \sum_{n=0}^{\infty} \lambda_n.P_n = \sum_{n=0}^{k-1} \lambda.P_n + \sum_{n=k}^{\infty} \lambda.e^{-\frac{n-k}{10}}.P_n$$

In steady state, we get: $W_q = \frac{L_q}{\lambda_{Av}} = \frac{\sum_{n=k}^{\infty}(n-k).P_n}{\sum_{n=0}^{k-1}\lambda.P_n + \sum_{n=k}^{\infty}\lambda.e^{-\frac{n-k}{10}}.P_n} = \frac{\sum_{m=0}^{\infty}m.P_{m+k}}{\lambda\left[\sum_{n=0}^{k-1}P_n + \sum_{m=0}^{\infty}e^{-\frac{m}{10}}.P_{m+k}\right]}$

c) **1.5 pt**
No, it's not. People would balked out do are not taken into account here. The quality of service will be overestimated.

d) **1.5 pt**

Percentage of people who balked: $P_{Bal} = 1 - \frac{\lambda_{Av}}{\lambda} = 1 - \left[\sum_{n=0}^{k-1}P_n + \sum_{m=0}^{\infty}e^{-\frac{m}{10}}.P_{m+k}\right]$