Introduction to Engineering Systems, ESD.00

System Dynamics

Lecture 4

Dr. Afreen Siddiqi

MIT ESD
Massachusetts Institute of Technology
Engineering Systems Division

Negative Feedback and Exponential Decay

Ref: Figure 8-6, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

- First-order linear negative feedback systems generate exponential decay
- The net outflow is proportional to the size of the stock
- The solution is given by: $S(t) = S_o e^{-dt}$
- Examples:

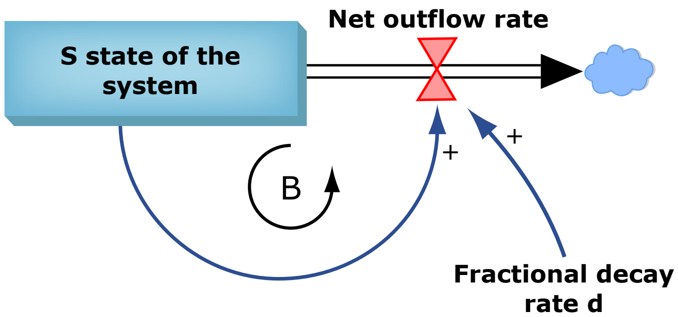


Image by MIT OpenCourseWare.

Net Inflow = -Net Outflow = -d*S

d: fractional decay rate [1/time]

Reciprocal of d is average lifetime units in stock.

Phase Plot for Exponential Decay

- In the phase-plot, the net rate of change is a straight line with negative slope

- The origin is a stable equilibrium, a minor perturbation in state S increases the decay rate to bring system back to zero – deviations from the equilibrium are self-correcting

- The goal in exponential decay is implicit and equal to zero

Negative Feedback with Explicit Goals

- In general, negative loops have non-zero goals

- Examples:

- The corrective action determining net flow to the state of the system is : Net Inflow = *f* (S, S*)

- Simplest formulation is:
  Net Inflow = Discrepancy/adjustment time = (S*-S)/AT

General Structure

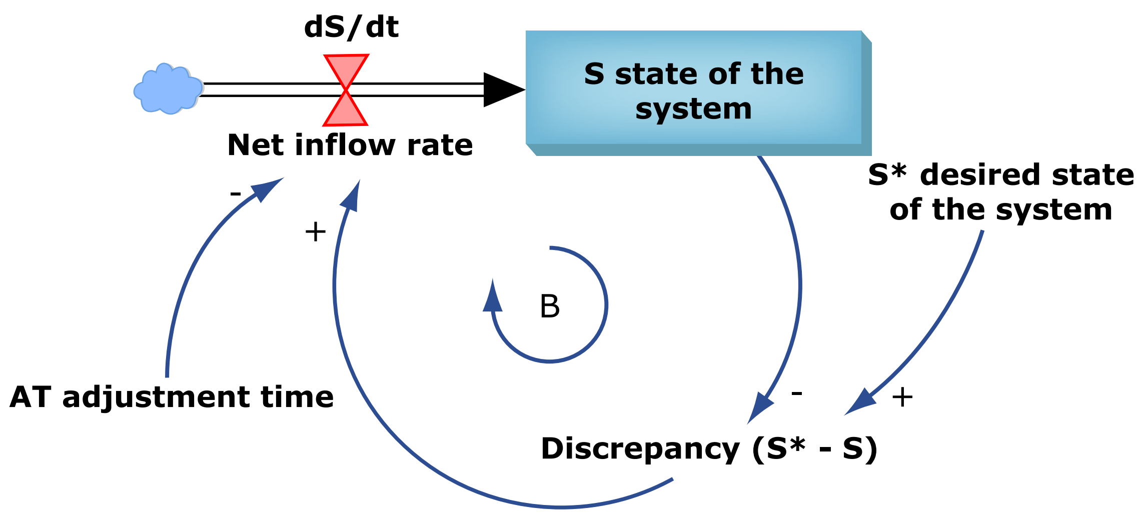


Image by MIT OpenCourseWare.

Ref: Figure 8-9, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

AT: adjustment time is also known as *time constant* for the loop

Phase Plot for Negative Feedback with Non-Zero Goal

- In the phase-plot, the net rate of change is a straight line with slope -1/AT

- The behavior of the negative loop with an explicit goal is also exponential decay, in which the state reaches equilibrium when S=S*

- If the initial state is less than the desired state, the net inflow is positive and the state increases (at a diminishing rate) until S=S*. If the initial state is greater than S*, the net inflow is negative and the state falls until it reaches S*

Time Constants and Settling Time

- For a first order, linear system with negative feedback, the system reaches 63% of its steady-state value in one time constant, and reaches 98% of its steady state value in 4 time constants.

- The steady state is not reached technically in finite time because the rate of adjustment keeps falling as the desired state is approached.

Time	Fraction of Initial Gap Remaining	Fraction of Initial Gap Corrected
0	$e^{-0} = 1$	$1-1 = 0$
τ	$e^{-1} = 0.37$	$1-e^{-1} = 0.63$
2τ	$e^{-2} = 0.14$	$1-e^{-2} = 0.87$
3τ	$e^{-3} = 0.05$	$1-e^{-3} = 0.95$
5τ	$e^{-5} = 0.007$	$1-e^{-5} = 0.993$

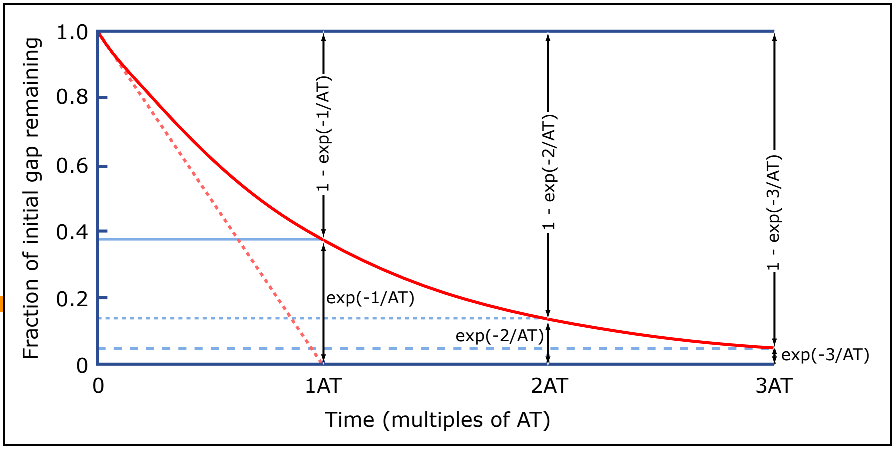


Image by MIT OpenCourseWare.

Ref: Figure 8-12, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

Multiple Loops

- Inflows and outflows are combined into a 'net rate'
- If birth rate (inflow) and death rate (outflow) are combined: Net Birth Rate = bP-dP
- dP/dt = (b-d)P
- P(t)=Po+integral[net birth rate]

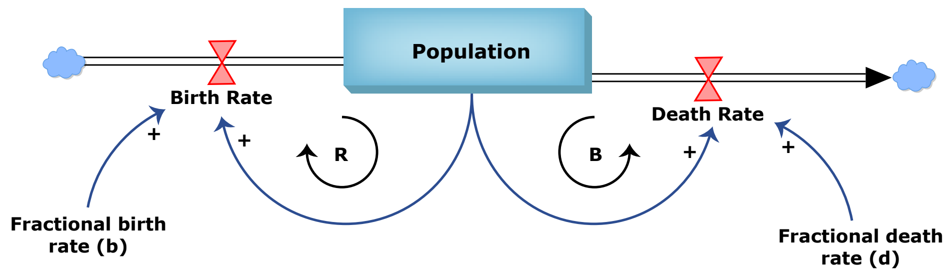


Image by MIT OpenCourseWare.

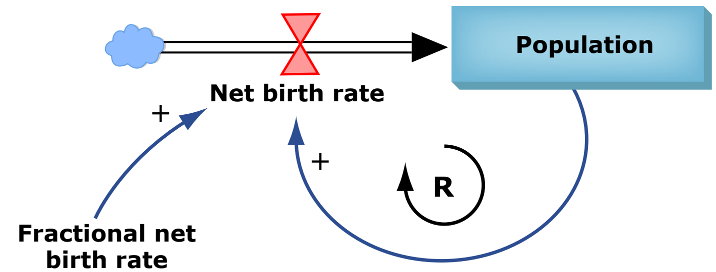


Image by MIT OpenCourseWare.

Ref: Figure 8-?, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

Linear Systems

- In a linear system the rate equations are weighted sums of state variables and any exogenous variable:

$$\frac{dS}{dt} = a_1 S_1 + a_2 S_2 + \cdots \; a_n S_n + b_1 U_1 + \cdots \; b_m U_m$$

- The super position property allows for summing the behavior of each individual loop to get overall behavior.

- Linear systems can be analyzed by reduction to their components. So linear systems, no matter how complex can be solved analytically to understand their dynamics.

- Realistic systems are far from linear.
- Linear system theory has dominated historically due to analytical tractability, but computers can now be readily used to simulate non-linear behavior

Non-Linear First Order Systems

- Population of real beings grow and stabilize, fluctuate or even collapse.
- The dominance of loops shifts over time – the behavior is non-linear
- In real systems, the fractional birth and death rates change as population approaches the carrying capacity.
- Carrying capacity is the population that can just be supported by the environment.
- Assume C is constant (neither consumed nor augmented), and model b and d to be functions of C.

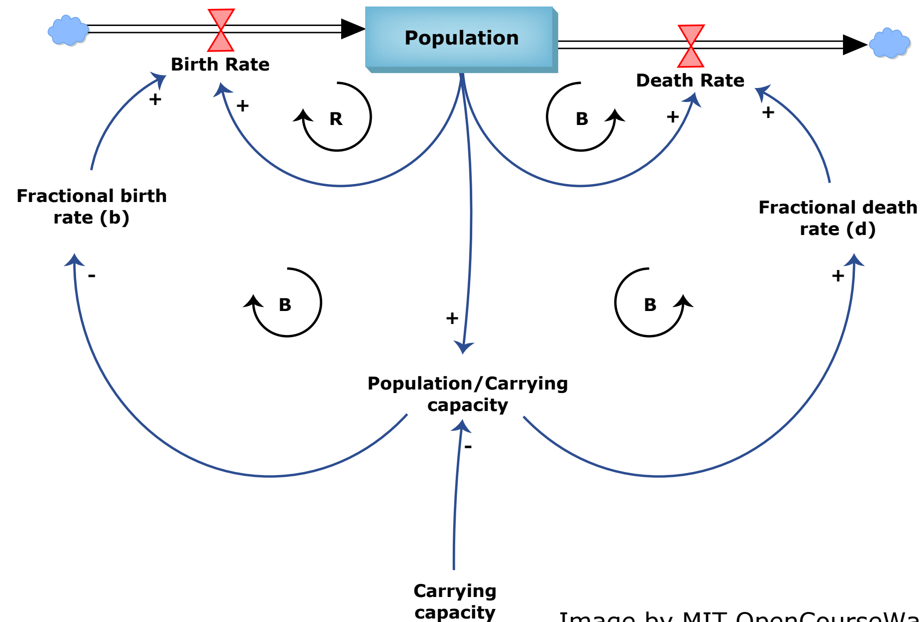


Image by MIT OpenCourseWare.

$$b = f_1\left(\frac{P}{C}\right)$$

$$d = f_2\left(\frac{P}{C}\right)$$

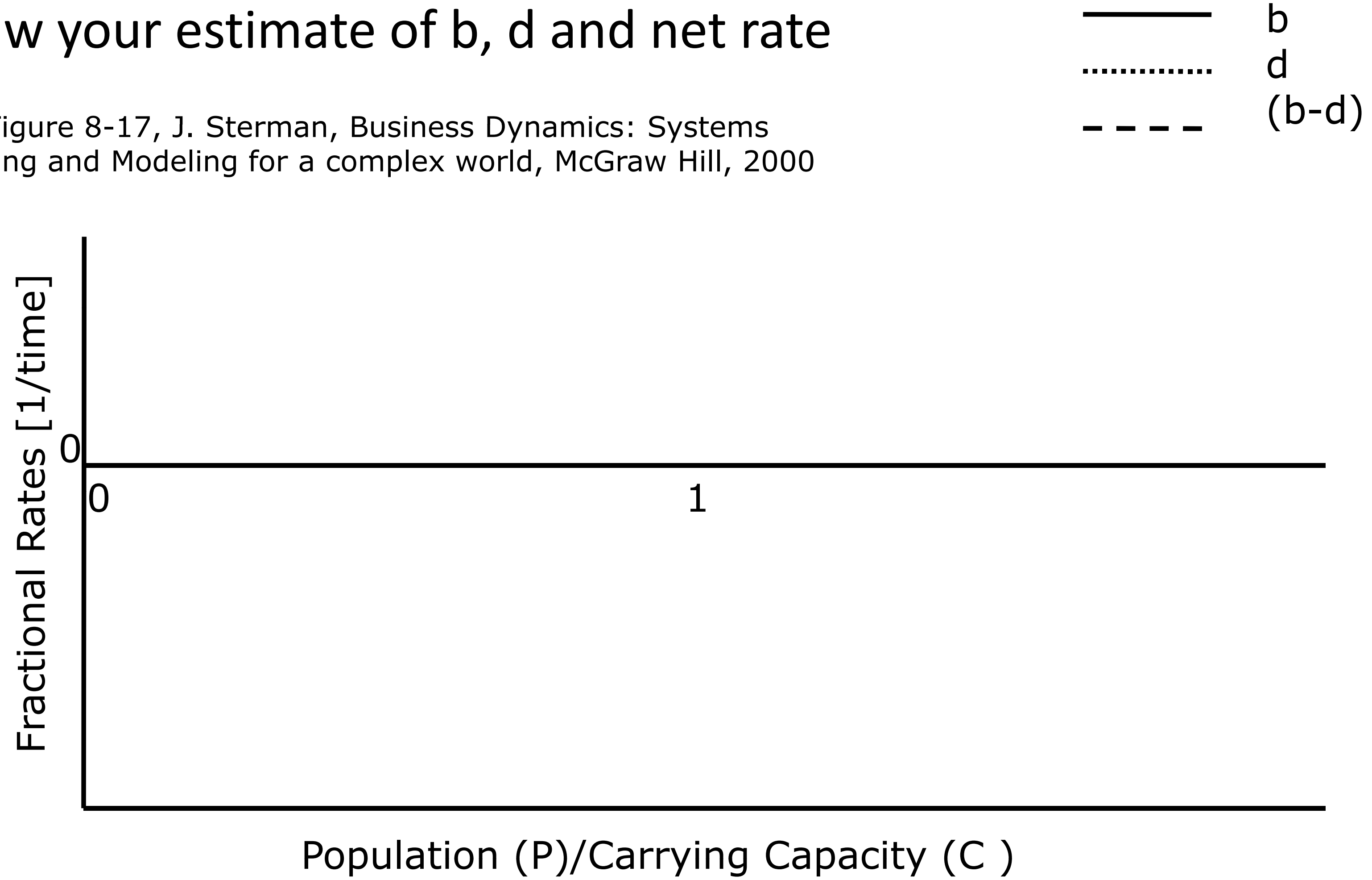
P: population
C: carrying capacity
b: fractional birth rate
d: fractional death rate

Ref: Figure 8-15, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

Non-Linear Rates:

- Draw your estimate of b, d and net rate

Ref: Figure 8-17, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000



Phase-Plots for Non-Linear Model

Time-Domain Plot for Population Growth

- Top Figure: Initially P<<C. Population follows an S shaped trajectory, with inflection point at (P/C $_{inf}$). Net birth rate is maximum at that point.
- Bottom Figure: Initially P>>C. P will decay until it reaches C.
- The P=C is a stable equilibrium point.

Ref: Figure 8-20, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

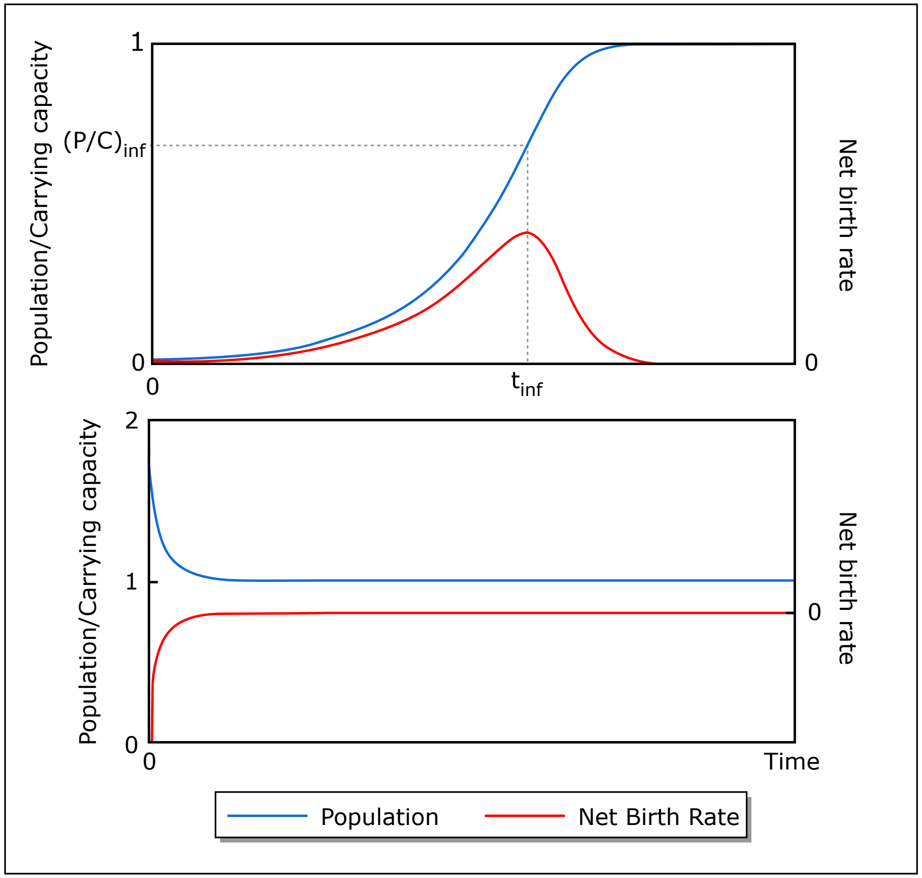


Image by MIT OpenCourseWare.

Logistic Growth

- A special case of S-shaped growth is known as logistic growth or Verhulst growth (first developed in 1938)

- In this model, the net fractional growth rate, *g*, is a linear function of the population.

$$g = g^{*}\left(1 - \frac{P}{C}\right)$$

g*: maximum fractional growth

- The net growth rate is then:

$$\frac{dP}{dt} = gP = g^{*}\left(1 - \frac{P}{C}\right)P$$

$$= g^{*}P - g^{*}\frac{P^{2}}{C}$$

Logistic Growth

- The logistic model can be represented in a non-linear analytic expression:

$$P(t) = \frac{C}{1 + \left[\frac{C}{P_0} - 1 \right] e^{-g^* t}}$$

- It has the property that maximum net growth rate occurs at exactly when population is at half the carrying capacity.

Spread of Infectious Disease: SI Model

Ref: Figure 9-4, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

Total Population: $N = S + I \quad [people]$

Infection rate: $I_R = \frac{dI}{dt} = Sc \cdot \frac{I}{N} \cdot i \quad [people/day]$

$$\frac{dI}{dt} = (N - I)c \cdot \frac{I}{N} \cdot i$$

$$\frac{dI}{dt} = ciI\left(1 - \frac{I}{N}\right)$$

$$\frac{dI}{dt} = ciI - ci\frac{I^2}{N}$$

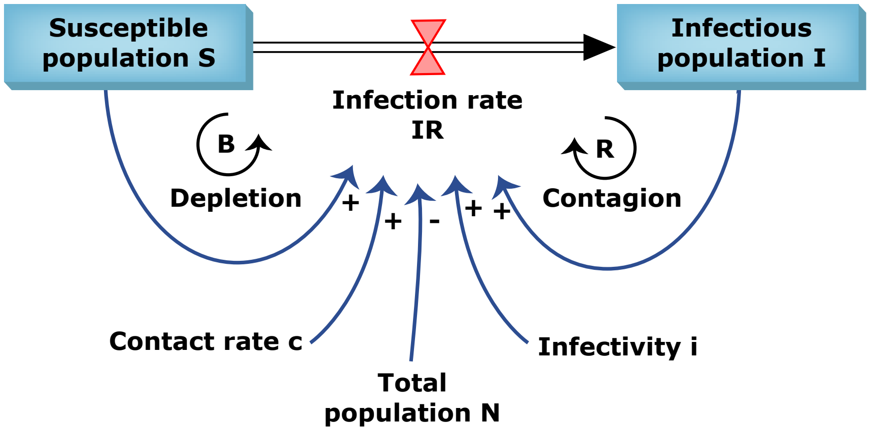


Image by MIT OpenCourseWare.

Key assumptions:

Total Population is constant (no migration, births, deaths etc)
No recovery – patients infected indefinitely
Constant contact rate

Key implication:
A single infected individual causes everyone in the community contract the disease

Spread of Infectious Disease: SIR Model

Ref: Figure 9-5, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

$$S = (N - I_o - R_0) + \int_{t_0}^{t_f} -I_R \, dt$$

$$I = I_o + \int_{t_0}^{t_f} (I_R - R_R) \, dt$$

$$R = R_o + \int_{t_0}^{t_f} R_R \, dt$$

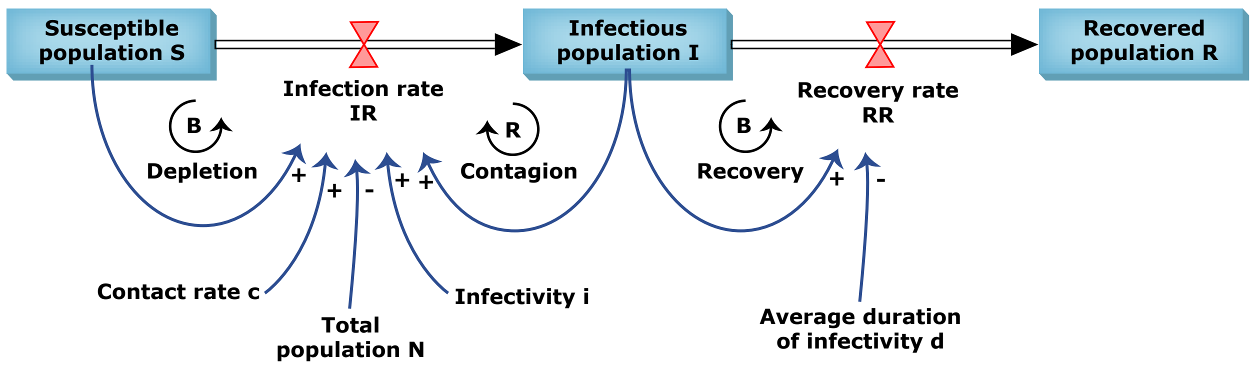


Image by MIT OpenCourseWare.

$$\text{Recovery rate: } R_R = \frac{I}{d} \quad [people/day]$$

d: average length of time people are infectious

Key assumption:
Patients remain sick for limited time then recover and develop immunity

Key implication:
Some people may not contract the disease

Greater the number of infected individuals, greater the recovery rate and then lower the total number of infected individuals – we get a balancing loop

The Tipping Point

- The SIR model is second-order (two independent stocks)

- Unlike the SI model, the disease may die out without causing an epidemic – if recovery rate is faster than infection rate, infectious population will fall causing infection rate to fall. I may go to zero before everyone catches the disease.

- When does an epidemic occur?

Path Dependence

- Path dependence is a pattern of behavior in which small chance events, early in the history of the system determine the ultimate end state, even when all end states are equally likely at the beginning.

- Path dependence arises in systems whose dynamics are dominated by positive feedback processes.

- A series of early random events essentially ‘lock’ the system into a particular equilibrium state. The theory of feedback and lock-in has been extensively researched for a variety of socio-technical systems in the context of business, technology and economics.

The Polya Process

- Consider a jar that is to be filled with stones – black stones and white stones
- Stones are added one at a time.
- The color of stone added to the jar each time is determined by chance.
- **The likelihood of selecting a black stone in the jar depends on the proportion of black stones already in the jar.**
- This rule makes the system path-dependent.
- This process is called the 'Polya' process, after its inventor George Polya.

The Polya Process

- Suppose the jar initially contains one black and one white stone.
- The likelihood of choosing a black stone in the next step is then proportion of black stones already present i.e. ½ (or 50% chance)
- Suppose, the next stone that gets picked does turn out to be black.
- Now the proportion of black stones is 2/3. The chance of the next stone that is picked being black is now 66.6%.
- Again, suppose that the next stone does turn out to be black. There are now four stones in the jar, and proportion of black stones is ¾, so the chance of the next stone being black is now 75%.
- Most likely, once we are through picking a number of stones and following the rule, the jar will mostly have black stones.
- **Now, think what would have happened if initially we had turned up a white stone instead….**
- The trajectory of the system and its end state depends on its early history

The Polya Process

- The Polya process contains two feedback loops, one positive and one negative for each type of stone.
- The greater the number of one type of stone, the greater the chance of adding another stone of that type (positive feedback).
- However, the greater the number of total stones, the smaller the impact of adding another stone of that type on total proportion of that type (negative feedback).

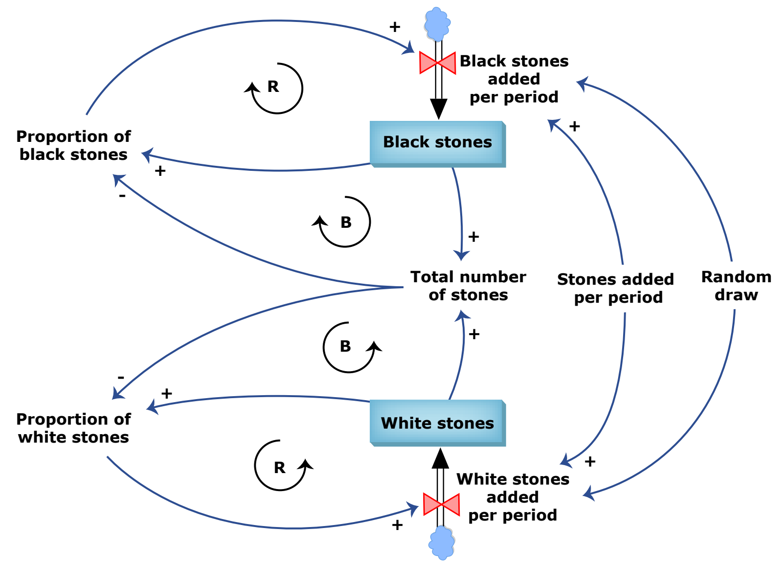


Image by MIT OpenCourseWare.

Rule for adding stones:

$$\text{Black stone added per period} = \begin{cases} 1 & \text{if proportion of black stones} > \text{random draw} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{White stone added per period} = \begin{cases} 1 & \text{if proportion of white stones} > \text{1-random draw} \\ 0 & \text{otherwise} \end{cases}$$

Ref: Figure 10-2, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

Ten Realizations of the Polya Process

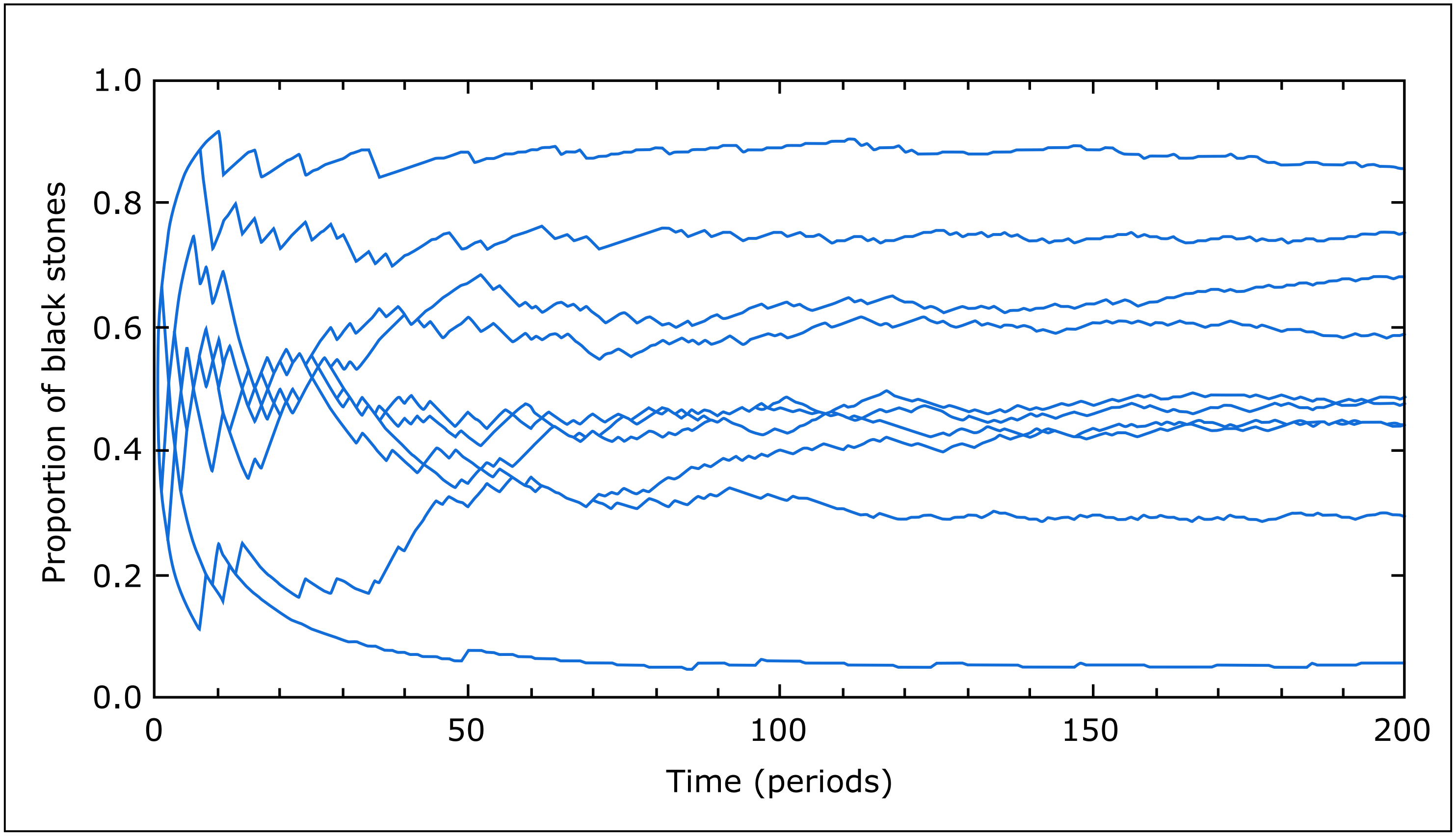


Image by MIT OpenCourseWare.

- Polya proved that the process always converges to a fixed proportion of black stones, and the particular proportion depends on early history.
- He also proved that **all proportions of black stones are equally likely** in the long run!

Ref: Figure 10-3, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

The Non-Linear Polya Process

- In general, the rules determining flow are non-linear functions.
- Suppose the likelihood of choosing a black stone is determined by a non-linear function such that when proportion of black stones rises above one-half, the likelihood of choosing a black stone rises by more than 50%, and if proportion is lower than one half, then the chance of choosing a black stone is much lower than 50%

Dynamics of the Nonlinear Polya Process

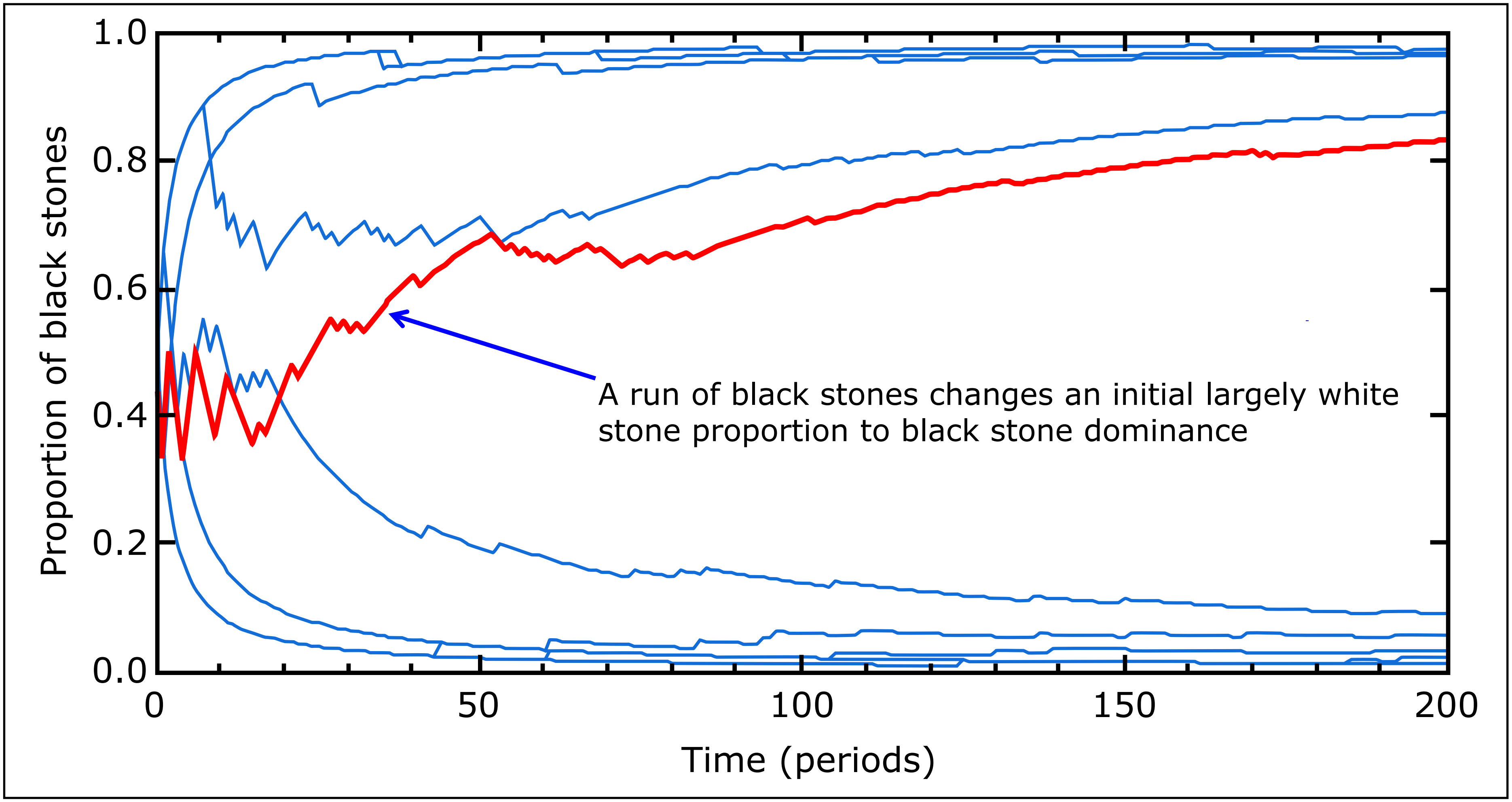


Image by MIT OpenCourseWare.

System tends toward all one color or all the other depending on early history-
Winner takes all!

Ref: Figure 10-7, J. Sterman, Business Dynamics: Systems Thinking and Modeling for a complex world, McGraw Hill, 2000

References

- *Business Dynamics: Systems Thinking and Modeling for a Complex World*, John D. Sterman, 2000, McGraw Hill

- Lecture 2: Chapters 1,3,5
- Lecture 3: Chapters 6,7,8
- Lecture 4: Chapters 8,9,10

ESD.00 Introduction to Engineering Systems
Spring 2011