# 6.857 Computer and Network Security
Lecture 14

Admin:
- Problem Set #4 out

Today:
- Malleability of El Gamal
- IND-CCA2 security (Cramer-Shoup claim)
- RSA
- Making RSA IND-CCA2 secure (OAEP)
- Other aspects of RSA security

**Theorem** (Tsiounis & Yung):

El Gamal is semantically secure in G

$\Leftrightarrow$

DDH holds in G

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- Semantic security may not be enough for some applications.
- El Gamal is malleable:

  Given $E(m) = (g^k, m \cdot y^k)$

  it is easy to produce $E(2m) = (g^k, (2 \cdot m) \cdot y^k)$

  without knowing $m$!

- More generally, El Gamal is homomorphic:

  Given $c_1 \in E(m_1) = (g^r, m_1 \cdot y^r)$

  & given $c_2 \in E(m_2) = (g^s, m_2 \cdot y^s)$

  can produce $c_1 \cdot c_2 = (g^{r+s}, (m_1 \cdot m_2) \cdot y^{r+s})$

  $\in E(m_1 \cdot m_2)$

- Product of ciphertexts yields an encryption of product of plaintexts.
- Special case: multiplying by $E(1) = (g^s, y^s)$ re-randomizes encryption.

- What is stronger notion of security for PK encryption? (e.g. one that excludes malleability...)
- "IND-CCA2 secure" (ACCA secure = secure under adaptive chosen ciphertext attack)
  ≈ IND-CCA secure defn we saw for symmetric enc.
- Similar to semantic security defn, except that Adv allowed access to decryption oracle, too.
  (He has PK so access to encryption oracle already there.)
  (As before, may <u>not</u> use oracle to decrypt challenge ciphertext during "guess" phase.)

## IND-CCA2 (ACCA) security game:

### Phase I ("Find"):

- Examiner generates $(PK, SK)$ using $Keygen(1^\lambda)$
- Examiner sends PK to Adversary
- Adversary computes for polynomial (in $\lambda$) time,
  having access to a decryption oracle $D(SK, \cdot)$
  then outputs two messages $m_0, m_1$ of same length,
  and "state information" s. $[m_0 \neq m_1$ required$]$

new ⇒

### Phase II ("Guess"):

- Examiner picks $b \xleftarrow{R} \{0,1\}$, computes $c_* = E(PK, m_b)$
- Examiner sends $c_*$, s to Adversary
- Adversary computes for polynomial (in $\lambda$) time,
  having access to a decryption oracle $D(SK, \cdot)$
  except on input $c_*$
  then outputs $\hat{b}$ (his "guess" for b).

new ⇒

Adversary wins if $\hat{b} = b$.

**Def:** PK encryption method is IND-CCA2 secure (ACCA-secure) if

$$\text{Prob}[\text{Adv wins}] \leq \frac{1}{2} + \text{negligible}$$

## How to make El Gamal IND-CCA2 secure?

- Cramer–Shoup method is such an extension of El Gamal.
- Let $G_q$ be a group of prime order $q$
  (e.g. $G_q = Q_p$, where $p = 2q+1$, $p$ & $q$ prime).
- Keygen:

$$g_1, g_2 \xleftarrow{R} G_q$$

$$x_1, x_2, y_1, y_2, z \xleftarrow{R} Z_q$$

$$c = g_1^{x_1} g_2^{x_2}$$

$$d = g_1^{y_1} g_2^{y_2}$$

$$\boxed{h = g_1^{z}}$$

EG

$$PK = (g_1, g_2, c, d, h)$$

$H$ = hash fn mapping $G_q^3$ to $Z_q$

$$SK = (x_1, x_2, y_1, y_2, z)$$

- **Enc(m) [where $m \in G_q$]:**

  $r \xleftarrow{R} Z_q$ EG

  $u_1 = g_1^r$ EG

  $u_2 = g_2^r$

  $e = h^r \cdot m$ EG

  $\alpha = H(u_1, u_2, e)$

  $v = c^r d^{r\alpha}$

  ciphertext $= (u_1, u_2, e, v)$ EG

- **Decrypt $(u_1, u_2, e, v)$:**

  $\alpha = H(u_1, u_2, e)$

  Check: $u_1^{x_1 + y_1 \alpha} u_2^{x_2 + y_2 \alpha} \stackrel{?}{=} v$

  If not equal, reject

  else output $m = e / u_1^z$ EG

Note: $u_1^{x_1} u_2^{x_2} = g_1^{r x_1} g_2^{r x_2} = c^r$

$u_1^{y_1} u_2^{y_2} = d^r$

$u_1^z = g_1^{rz} = h^r$ EG

**Theorem:** Cramer-Shoup is IND-CCA2 secure (i.e. secure against adaptive chosen ciphertexts) if

① DDH holds in $G_q$

② H satisfies a certain condition (≈ "target collision resistance")

Thus, our strongest notion of security for PK encryption is in fact achievable, albeit at some cost in terms of speed & complexity.

## Diffie-Hellman model of PK encryption: (1976)

- Keygen $(1^\lambda) \rightarrow (PK, SK, M, C)$

  (public key, secret key, message space, ciphertext space)

  Here $|M| = |C|$.

- $E(PK, \cdot)$ is an efficiently computable one-to-one (deterministic) map from M to C

  $c = E(PK, m)$ is (unique) ciphertext for m

- $D(SK, \cdot)$ is efficiently computable inverse:

  $D(SK, c) = D(SK, E(PK, m)) = m \quad (\forall m \in M)$

- It is hard/infeasible to decrypt with knowledge of PK but without knowledge of SK.

  SK represents "trapdoor" information that enables inversion of the (otherwise one-way) function $E(PK, \cdot)$.

## RSA PK encryption (Rivest, Shamir, Adleman 1977)

**Keygen:**

Find two large primes $p, q$ (e.g. $\lambda = 1024$ bits each)

$n = p \cdot q$

$\varphi(n) = |Z_n^*| = (p-1)(q-1)$

$e \xleftarrow{R} Z_{\varphi(n)}^*$ [i.e. $\gcd(e, \varphi(n)) = 1$]

$d = e^{-1} \pmod{\varphi(n)}$ [e.g. Euclid's extended alg]

$PK = (n, e)$

$SK = (d, p, q)$

$M = C = Z_n$

**Encrypt:**

Given $m \in Z_n$ and $PK = (n, e)$:

$$c = E(PK, m) = m^e \pmod{n}$$

**Decryption:**

Given $c \in Z_n$ and $SK = (d, p, q)$:

$$m = D(SK, c) = c^d \pmod{n}$$

(where $n = p \cdot q$)

**Note:**

$p$ & $q$ should be large randomly chosen primes, as security of RSA depends upon inability of adversary to factor $n$ (from PK) into $p, q$.

## Correctness of RSA:

**Lemma:** (Chinese remainder theorem or CRT)

Let $n = p \cdot q$ where $p, q$ are distinct primes.

Then $(\forall x, y \in Z_n)$

$$x = y \pmod{n} \iff x = y \pmod{p} \;\&\; x = y \pmod{q}$$

Thus it suffices to prove RSA correct mod $p$; the proof mod $q$ is the same, and CRT then implies correctness mod $n$.

Given $e \cdot d = 1 \pmod{\varphi(n)}$ $\quad [d = e^{-1} \pmod{\varphi(n)}]$

so $e \cdot d = 1 + t \cdot (p-1)(q-1)$ for some $t$

and $e \cdot d = 1 \pmod{(p-1)}$ $\quad [d = e^{-1} \pmod{p-1}]$

Correctness of RSA means

$$(m^e)^d = m \pmod{n} \quad \text{for all } m \in Z_n$$

By CRT we only need to prove

$$(m^e)^d = m \pmod{p} \quad \text{for all } m \in Z_p$$

We consider two cases:

**Case 1:** $m = 0 \pmod p$

Trivial: $0^{ed} = 0 \pmod p$

**Case 2:** $m \neq 0 \pmod p$

i.e. $m \in Z_p^*$

so $m^{p-1} = 1 \pmod p$ [Fermat]

Then $m^{ed} = m^{1+u\cdot(p-1)} \pmod p$

where $u = t\cdot(q-1)$

$$m^{ed} = m \cdot (m^{p-1})^u \pmod p$$
$$= m \cdot 1^u$$
$$= m$$

$\therefore$ $m^{ed} = m \pmod p$ for all $m \in Z_p$

& $m^{ed} = m \pmod q$ for all $m \in Z_q$ (similarly)

and $m^{ed} = m \pmod n$ for all $m \in Z_n$ (by CRT)

Thus $(\forall m \in Z_n)\ D(SK, E(PK, m)) = m$ ▨

# Security of RSA

## Factoring attacks:

If any adversary can factor $n$, then the adversary can compute $\varphi(n)$, and thus compute $d = e^{-1} \pmod{\varphi(n)}$. & vice versa

Key insight: size of group $Z_n^*$ is unknown and unknowable to Adversary.

## How hard is factoring?

- time $\exp\{c \cdot (\ln n)^{1/3} (\ln \ln n)^{2/3}\}$
- RSA keys of length 768 factored (2009); can expect RSA keys of length 1024 bits to be factored in the "near future".
- RSA keys of length 2048 secure for a very long time, unless there are algorithmic breakthroughs on problem of factoring.

## Is (basic) RSA semantically secure?

No. (It's not even randomized...)

∴ not IND-CCA2 secure either...

## How to make RSA IND-CCA2 secure?

OAEP = "Optimal asymmetric encryption padding" [BR 94]

Let message $m$ be $t$ bits in length.
Add $k_0$ bits of randomness $|r| = k_0$
Add $k_1$ bits of 0's $0^{k_1}$ (to check)

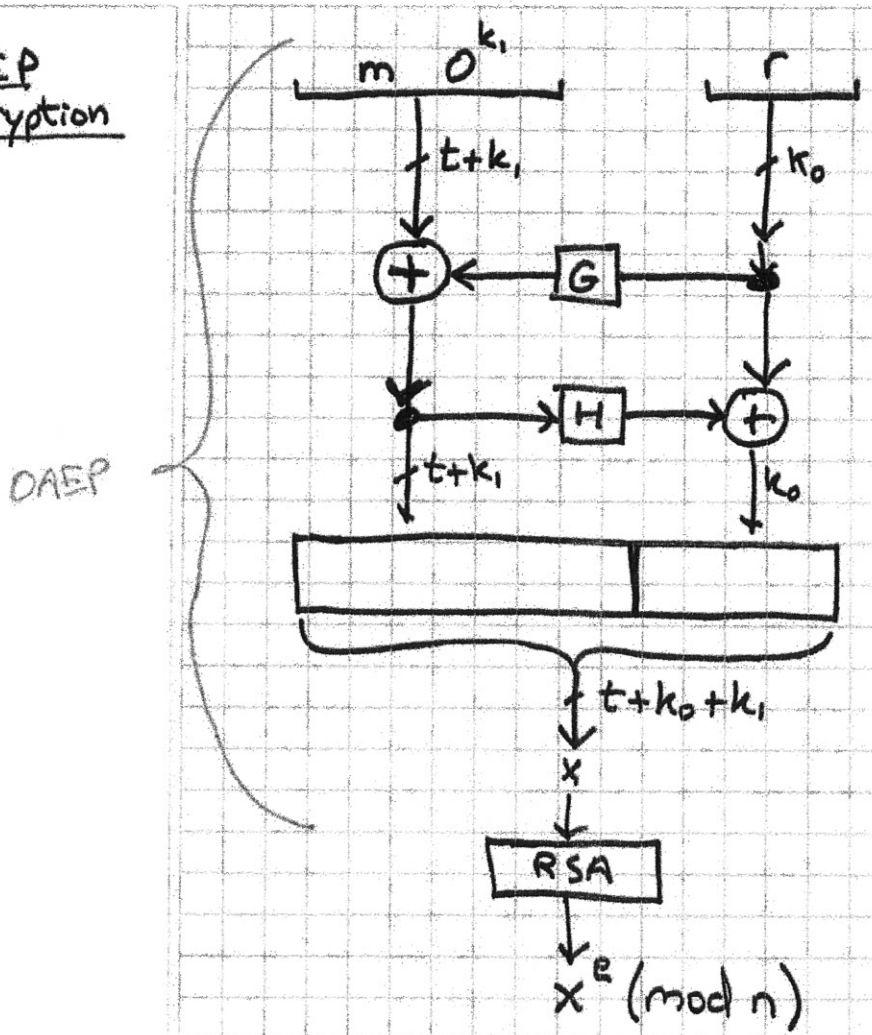
Assume $G: \{0,1\}^{k_0} \to \{0,1\}^{t+k_1}$

$H: \{0,1\}^{t+k_1} \to \{0,1\}^{k_0}$

G, H "random oracles"

[Compare to UFE of Desai for symmetric encryption]

## OAEP Encryption




On decryption:
- invert RSA
- invert OAEP
- reject if $0^{k_1}$ not present
- else output m

**Theorem:** RSA with OAEP is IND-CCA2 secure, assuming ROM for G & H, and assuming RSA hard to invert on random inputs.

[Bug in original proof, but OK with very slightly modified assumptions (or OAEP+)]

OAEP used in practice

(But in practice we don't really have random oracles!)

## Other aspects of RSA security:

[ref Boneh paper: 20 years of attacks on RSA]

**Weak keys:** small $d$ is insecure

($d < n^{1/4}$ allows adversary to factor $n$)

**Implementation issues:**

- Power analysis } "side channel attacks"
- Timing attacks }
- Fault injection (introduce power supply glitch)

(esp. if device is using CRT)

**Quantum computing**

Peter Shor (MIT) has shown that factoring in polynomial time is possible on a "quantum computer"

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Spring 2014