6.851: Advanced Data Structures Spring 2012

Lecture 14 — April 10, 2012

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1 Overview

Today we are going to go over:

- The reduction between sorting and priority queues
- A survey of sorts
- Bitonic Sequences
- Logarithmic Merge Operation
- Packed Sorting
- Signature Sort Algorithm

The reduction and survey will serve as motivation for signature sort. Bitonic Sequences will be used to build Logarithmic Merge which will in turn be used to build Packed Sorting which will in finally be used to build Signature Sort.

2 Sorting reduced to Priority Queues

Thorup [7] showed that if we can sort n w-bit integers in $O(nS(n, w))$, then we have a priority queue that can support the insertion, deletion, and find minimum operations in $O(S(n, w))$. To get a constant time priority queue, we need linear time sorting.
OPEN: linear time sorting

3 Sorting reduced to Priority Queues

Following is a list of results outlining the current progress on this problem.

- Comparison model: $O(n \lg n)$
- Counting sort: $O(n + 2^w)$
- Radix sort: $O(n \cdot \frac{w}{\lg n})$

- van Emde Boas: $O(n \lg w)$, improved to $O(n \lg \frac{w}{\lg n})$ (see [6]).
- Signature sort: linear when $w = \Omega(\lg^{2+\varepsilon} n)$ (see [2]).
- Han [4]: $O(n \lg \lg n)$ deterministic, AC^0 RAM.
- Han and Thorup: $O(n\sqrt{\lg \lg n})$ randomized, improved to $O(n\sqrt{\lg \frac{w}{\lg n}})$ (see [5] and [6])).

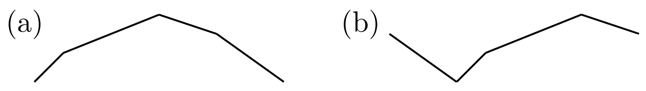
Today, we will focus entirely on the details of the signature sort. This algorithm works whenever $w = \Omega(\log^{2+\varepsilon} n)$. Radix sort, which we should already know, works for smaller values of w, namely when $w = O(\log n)$. For all other values of w and n, it is open whether we can sort in linear time. We previously covered the van Emde Boas tree, which allows for $O(n \log \log n)$ sorting whenever $w = \log^{O(1)} n$. The best we have done in the general case is a randomized algorithm in $O(n\sqrt{\log \frac{w}{\log n}})$ time by Han, Thorup, Kirkpatrick, and Reisch.
OPEN: $O(n)$ time sort $\forall w$

4 Sorting for $w = \Omega(\log^{2+\varepsilon} n)$

The signature sort was developed in 1998 by Andersson, Hagerup, Nilsson, and Raman [2]. It sorts n w-bit integers in $O(n)$ time when $w = \Omega(\log^{2+\varepsilon} n)$ for some $\varepsilon > 0$. This is a pretty complicated sort, so we will build the algorithm from the ground up. First, we give an algorithm for sorting bitonic sequences using methods from parallel computing. Second, we show how to merge two words of $k \leq \log n \log \log n$ elements in $O(\log k)$ time. Third, using this merge algorithm, we create a variant of MERGESORT called *packed sorting*, which sorts n b-bit integers in $O(n)$ time when $w \geq 2(b+1) \log n \log \log n$. Fourth, we use our packed sorting algorithm to build signature sort.

4.1 Bitonic Sequences

A *bitonic sequence* is a cyclic shift of a monotonically increasing sequence followed by a monotonically decreasing sequence. When examined cyclically, it has only one local minimum and one local maximum.



Cyclic shifts preserve the binoticity of a sequence.

To sort a bitonic sequence `btcseq`, we run the algorithm shown below in the figure below. Assume $n = \mathrm{len(btcseq)}$ is even.

Bitonic sequence sorting maintains two invariants: (a) the sequence `btcseq` contains multiple sections of bitonic sequences (each level of `btcsort` splits a bitonic sequence into two bitonic sequences), and (b) when considering two sections of bitonic sequences, an element in the left section is always smaller than an element in the right section.

Algorithm 1 Bitonic sequence sorting algorithm

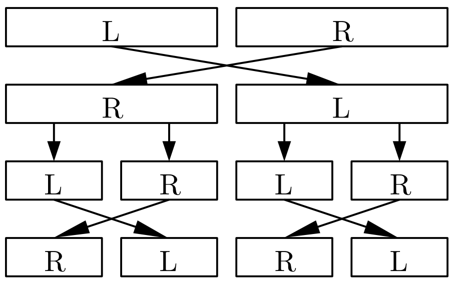
```
btcsort(btcseq):
  for i from 0 to n/2-1:
    if btcseq[i]>btcseq[i+n/2]:
      swap(btcseq[i], btcseq[i+n/2])
  btcsort(btcseq[0:n/2-1])
  btcsort(btcseq[n/2:n-1])
```

After $O(\log n)$ rounds, `btcseq` is broken into n sections. Since the left section has elements smaller than the right section, sorting is complete.

For more information about sorting bitonic sequences, including a proof of the correctness of this algorithm, see [3, Section 27.3].

4.2 Logarithmic Merge Operation

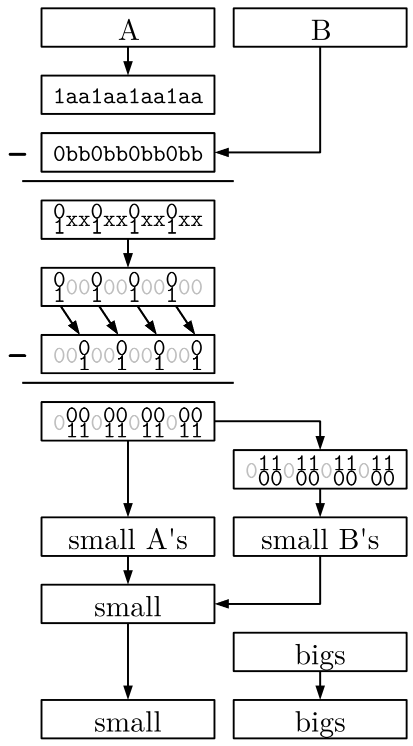
The next step is to merge two sorted words, each containing k b-bit elements. First, we concatenate the first word with the reverse of the second word, getting a bitonic sequence. To efficiently reverse a word, we mask out the leftmost $\frac{k}{2}$ elements and shift them right by $\frac{k}{2}b$, then mask out the rightmost $\frac{k}{2}$ elements and shift them left by $\frac{k}{2}b$. Taking the OR of the two resulting words leaves us with the original word with the left and right halves swapped. We can now recurse on the left and right halves of the word, giving us the recursion $T(n) = T(\frac{k}{2}) + O(1)$, so the whole algorithm takes $T(k) = O(\log k)$ time. The two words may now be concatenated by shifting the first word left by kb and taking its OR with the second word. The key here is to perform each level of the recursion in parallel, so that each level takes the same amount of time. This list reversal is illustrated in the figure below.



The first two steps in the recursion for reversing a list.

All that remains is to run the bitonic sorting algorithm on the elements in our new word. To do so, we must divide the elements in two halves and swap corresponding pairs of elements which are out of order. Then we can recurse on the first and second halves in parallel, once again giving us the recursion $T(k) = 2T(\frac{k}{2}) + O(1) \Rightarrow T(k) = O(\log k)$ time. Thus we need a constant-time operation which will perform the desired swapping.

Assume we have an extra 0 bit before each element packed into the word. We use this spare bit to help bitmask our word. We will mask the left half of the elements and set this extra bit to 1 for each element, then mask the right half of the elements and shift them left by $\frac{k}{2}b$. After we subtract the second word from the first, a 1 will appear in the extra bit iff the element in the corresponding position of the left half is greater than the element in the right half. Thus we can mask the extra bits, shift the word right by $b-1$ bits, and subtract it from itself, resulting in a word which will mask all the elements of the right half which belong in the left half and vice versa. From this, we use bit shifts, OR, and negation to get our sorted list. See the diagram below for clarification; the process is fairly straightforward.



Parallel swap operation in bitonic sorting.

Using these bit tricks, we end up with our desired constant time all-swap operation. This leads to the following theorem:

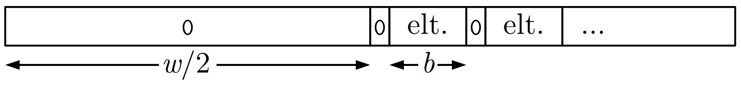
Theorem 1. *Suppose two words each contain k b-bit elements. Then we can merge these words in $O(\log k)$ time.*

Proof. As already noted, we can concatenate the first word with the reverse of the second word in $O(\log k)$ time. We can then apply the bitonic sorting algorithm to these concatenated words. Since this algorithm has a recursion depth of $O(\log k)$, and we can perform the required swaps in constant time, our merge operation ends up with time complexity $O(\log k)$. □

4.3 Packed Sorting

In this section we will present packed sorting [1], which sorts n b-bit integers in $O(n)$ time for a word size of $w \geq 2(b+1)\log n \log\log n$. This bound for w allows us to pack $k = \log n \log\log n$

elements into one word, leaving a zero bit in front of each integer, and $\frac{w}{2}$ zero bits at the beginning of the word.



Structure for packing b-bit integers into a w-bit word.

To sort these elements using packed sorting, we build up merge operations as follows:

1. Merge two sorted words in $O(\lg k)$ time, as shown with bitonic sorting in the previous section.
2. Merge sort on k elements with (1) as the merge operation. This sort has the recursion $T(k) = 2T(\frac{k}{2}) + O(\lg k) \Rightarrow T(k) = O(k)$.
3. Merge two sorted lists of r sorted words into one sorted list of $2r$ sorted words. This is done in the same manner as in a standard merge sort, except we use (1) to speed up the operation. We start by merging the left-most words in the two lists using (1). The first word following this merger must consist of the smallest k elements overall, so we output this word. However, we cannot be sure about the relative position of the elements in the second word. We therefore examine the maximum element in the second word, and add the second word to the beginning of the list which contained this maximum element. We continue merging the lists in this manner, resulting in $O(r \log k)$ overall.
4. Merge sort all of the words with (3) as the merge operation and (2) as the base case.

Theorem 2. *Let word size $w \geq 2(b+1)\log n \log\log n$. Then packed sorting sorts n b-bit integers in $O(n)$ time.*

Proof. The sort defined in Step 4 of the algorithm above has the recursion $T(n) = 2T(\frac{n}{2}) + O(\frac{n}{k}\log k)$ and the base case $T(k) = O(k)$. The recursion depth is $O(\log \frac{n}{k})$, and each level takes $O(\frac{n}{k}\lg k) = O(\frac{n}{\lg n})$ time, so altogether they contribute a cost of $O(n)$. There are also $\frac{n}{k}$ base cases, each taking $O(k)$ time, giving a total runtime of $O(n)$, as desired. □

4.4 Signature Sort Algorithm

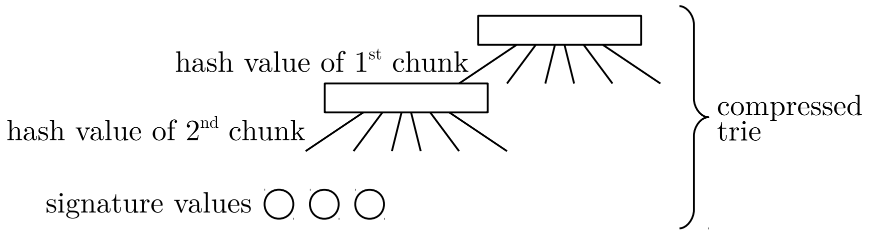
We use our packed sorting algorithm to build our seven step signature sort. We assume that $w \geq \log^{2+\varepsilon} n \log\log n$. The signature sort will sort n w-bit integers in $O(n)$ time. We will start by breaking each integer into $\lg^{\varepsilon} n$ equal-size chunks, which will then be replaced by $O(\lg n)$-bit signatures via hashing. These smaller signatures can be sorted with packed sorting in linear time. Because hashing does not preserve order, we will build a compressed trie of these sorted signatures, which we will then sort according to the values of the original chunks. This will allow us to recover the sorted order of the integers. The signature sort will proceed as follows:

1. Break each integer into $\lg^{\varepsilon} n$ equal-size chunks. (Note the distinguishment from a fusion tree, which has chunks of size $\lg^{\varepsilon} n$)

2. Replace each chunk by a $O(\lg n)$-bit hash (static perfect hashing is fine). By doing this, we end up with n $O(\lg^{1+\varepsilon} n)$-bit *signatures.* One way we can hash is to multiply by some random value x, and then mask out the hash keys. This will allow us to hash in linear time. Now, our hash does not preserve order, but the important thing is that it does preserve identity.

3. Sort the signatures in linear time with packed sorting, shown above.

4. Now we want to rescue the identities of the signatures. Build a compressed trie over the signatures, so that an inorder traversal of the trie gives us our signatures in sorted order. The compressed trie only uses $O(n)$ space.



A trie for storing signatures. Edge represent hash values of chunks; leaves represent possible signature values.

To do this in linear time, we add the signatures in order from left to right. Since we are in the word RAM, we can compute the LCP with $(i-1)^{\text{st}}$ signature by taking the most significant 1 bit of the XOR. Then, we walk up the tree to the appropriate node, and charge the walk to the decrease in the rightmost path length of the trie. The creation of the new branch is constant time, so we get linear time overall. This process is similar to the creation of a Cartesian tree.

5. Recursively sort the edges of each node in the trie based on their actual values. This is a recursion on (node ID, actual chunk, edge index), which takes up $(O(\log n), O(\frac{w}{\log^{\varepsilon} n}), O(\log n))$ space. The edge indices are in there to keep track of the permutation. After a constant $\frac{1}{\varepsilon}+1$ levels of recursion, we will have $b = O(\log n + \frac{w}{\log^{1+\varepsilon} n}) = O(\frac{w}{\log n \log\log n})$, so we can use packed sorting as the base case of the recursion.

6. Permute the child edges.

7. Do an inorder traversal of the trie to get the desired sorted list from the leaves.

Putting these steps together, we get:

Theorem 3. *Let $w \geq \log^{2+\varepsilon} n \log\log n$. Then signature sort will sort n w-bit integers in $O(n)$ time.*

Proof. Breaking the integers into chunks and hashing them (Steps 1 and 2) takes linear time. Sorting these hashed chunks using packed sort (Step 3) also takes linear time. Building the compressed trie over signatures takes linear time (Step 4), and sorting the edges of each node (Step 5) takes constant time per node for a linear number of nodes. Finally, scanning and permuting the nodes (Step 6) and the in-order traversal of the leaves of the trie (Step 7) will each take linear time, completing the proof. □

References

[1] Susanne Albers, Torben Hagerup: *Improved Parallel Integer Sorting without Concurrent Writing*, Inf. Comput. 136(1): 25-51, 1997.

[2] A. Andersson, T. Hagerup, S. Nilsson, R. Raman, *Sorting in Linear Time?*, J. Comput. Syst. Sci. 57(1): 74-93, 1998.

[3] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein: *Introduction to Algorithms*, Second Edition, The MIT Press and McGraw-Hill Book Company 2001.

[4] Y. Han: *Deterministic Sorting in $O(n \log \log n)$ Time and Linear Space*, J. Algorithms 50(1): 96-105, 2004.

[5] Y. Han, M. Thorup: *Integer Sorting in $O(n\sqrt{\log \log n})$ Expected Time and Linear Space*, FOCS 2002: 135-144.

[6] D.G. Kirkpatrick, S. Reisch: *Upper Bounds for Sorting Integers on Random Access Machines*, Theoretical Computer Science 28: 263-276 (1984).

[7] M. Thorup: *Equivalence between Priority Queues and Sorting.* FOCS 2002: 125-134 (2002).