PROFESSOR: All right, welcome. We have some fun origami topics today, mostly related to a topic called pleat folding. General idea of pleat folding is to fold alternating mountains and valleys back and forth. And you can do a lot of cool things with pleat folding, and that's what I want to talk about today.

In particular, this model is usually called the hyperbolic paraboloid. It's pretty simple. You start with a square paper. You fold this crease pattern, which is concentric squares alternating mountain and valley, and you fold the diagonals, alternating mountain and valley.

You crush it all together, you'll get this sort of x-shape. And you sort of let go, give it a twist, and it pops into this saddle form, which looks a lot like a hyperbolic paraboloid, which is one of the standard saddle surfaces. It's approximating a hyperbolic paraboloid, we think, or we thought, and that's why it's called that.

So you should all try this at home. It's a great model. It's very cool. It's an example of something I like to call self-folding origami because the paper basically folds itself. You put in a very simple crease pattern, concentric squares. What could be simpler than that? And yet you get this really cool 3D form automatically. Physics, in some sense, is finding this form for us.

You can take this pleating idea and apply it to many different crease patterns. Another fun one if you have a compass lying around, to score circles is you just crease concentric circles, alternating mountain and valley. In this case, you also need to cut out a circular hole in the center. Then you get a different saddle form, which we don't know exactly what it is.

Let's see, I have some various fun things to talk about here. So what's going on with paper-- if I take a piece of paper, paper is an elastic material. It remembers that it was made flat. It likes to be flat. If I curl a paper a little bit, for example, it flattens back out. That's elastic memory.

If however, I crease a piece of paper, then that's called a plastic deformation beyond the yield point of the material. This works in sheet metal also, lots of materials. I've effectively changed the memory of the paper. It now wants to stay bent. If I try to unfold it, it goes back to some angle. Depending on how hard I crease it, it will go to a sharper angle.

So that's basically what's going on. In these surfaces, where we crease the paper, it wants to stay bent. Where we don't crease the paper, it wants to stay flat. It can't do all of those things, but physics finds an equilibrium among all those forces.

So we simulated that some years ago. This is with architecture undergrad Jenna Fizel and Professor John Ochsendorf, architecture. So on the right of each of these examples, we have photographs of real models, pleated foldings, a hyperbolic paraboloid, squares, hexagon, octagon, just not so pretty.

But we re-created that physical model with a virtual model just using some spring approximation to the forces I described. And here's an approximation to a circle, where we just took a really big regular n-gon. This is real circles and the simulation.

So this confirms that what I said is actually true. Those are the only forces you need to model, and you get an approximation of what really happens in real life. So instead of folding real paper in this way, you could simulate it. Cool.

Here is that simulator in action with pleated hexagons, just to show you what it looks like. So the creases are trying to become more bent, at this point have reached equilibrium, and you get your 3D form. Once you have that 3D form, so you have a virtual model of a physical piece of paper, the natural thing to do is build a physical model of the virtual model of the physical piece of paper.

And you've seen this on the mailing list. This is what we're calling origami skeletons. Here the vertices, which you can see bigger here, are 3D printed plastic spheres, and there's aluminum rods. The spheres have holes at just the right angles to make exactly the 3D form that was made by that simulator. So this is something you could really only build if you had a virtual model of that thing.

So this is a fun example of taking mathematics and our sort of the computational understanding of how paper folds and turning it into sculpture that requires that mathematical basis.

On the topic of sculpture, at the very beginning-- so this is early on in my PhD, with my dad and my PhD advisor Anna Lubiw, same authors as like the folding problem, one of our early explorations on the mathematical sculpture side was to take hyperbolic paraboloids and take a lot of them and join them together to make polyhedral surfaces. These are what we call hyperhedra because hyperbolic paraboloids are also called hypars, I think, originally by the architect community.

So these are algorithmically-generated sculpture. The input to the algorithm is a polyhedron, like the cube, and the output is a way to join hypars together to represent that cube. And the algorithm is very simple.

For each face of the polyhedron-- so here it has four sides, you take four hyperbolic paraboloids and join them together in a cycle just sharing one edge. So you do that for each of the six faces, so in all you have 24 hyperbolic paraboloids.

Then to join two faces together, you join them along a pair of edges like this. And when you do that, you get an enclosure. It's a closed solid. And you can do that for every-- here we're doing it for all the Platonic solids. You can do it with any polyhedron in theory. So it's like an infinite family of sculptures here, which is kind of fun. Here is a simulation of that cube just for fun. That works, too.

And back to the curve creases. These are some examples I showed way back in lecture 1. These are sculptures in the permanent collection at MOMA. And the idea here is instead of taking just one concentric circle which goes around 360 degrees, we take a circular ramp which goes around twice, and then join the ends. So we have 720 degrees of material there, which you might think of as negative curvature.

But it's a little weird because there's the hole cut out in the center. So it's a little hard to measure curvature, but you can actually define it. And these are three different foldings with the same idea, slightly different parameters in how big those ramps are, and different numbers of creases. And you get very different equilibrium forms. Again, this all sort of self-folding paper wants to live in these three configurations.

Here's some that you may not have seen. They're on the web. But these are taking regular 360-degree circles but taking two or three or four of them and joining them at a couple of key points, and the rest is self-folding.

So the big-- I mean, the sort of powerful scientific engineering idea here is that you could deploy complicated 3D structures just by manufacturing very simple flat structures, maybe joining a few points together, and then just say go. If you can manufacture these things so that every crease locally wants to bend, then you'll get these 3D forms automatically.

This is especially powerful at the nano scale, where you can't have your fingers moving around and pressing things. But you can probably use materials that, say, when heated or you add some chemical, cause everything to fold. It's very easy for us from all the chip fabrication we do to manufacture flat crease patterns, and that will let us manufacture 3D things at the nano scale.

You could also imagine it a much larger scale like a space station-- like a space station that looks like that. That would be pretty cool-- and something where you don't want to have to physically fold everything by hand. But if it could be done automatically, life would be good.

Now, we don't have any algorithms for the reverse engineering problem-- If I give you a 3D curved surface like this, find the crease pattern and the joins that make it happen. But that's the goal. And towards that goal, we make sculpture to explore the space of what you can make.

All right. I think I have one more sculptural example here, which is combining these curve creases with glassblowing. And to make glassblowing more tactical, Marty here, our cameraman, is blowing glass blindfold. Don't try this at home.

So paper folding is a very tactile experience. To make glassblowing more about touching the material, which you're not usually supposed to do because it's over

1,000 degrees Fahrenheit-- blindfolds.

AUDIENCE: 1,000 degrees Centigrade.

PROFESSOR: 1,000 degrees centigrade. Wow. The temperature of an erupting volcano.

[MUSIC PLAYING]

This is in the glassblowing studio in 4.003 near here. He's made a whole bunch of those. And here's what it looks like to fold from scratch, so to speak, a concentric circle model. Of course, it's accelerated in movie time. It only takes 10 seconds. In reality, it takes 10 minutes after pre-scoring, maybe more. And now we get some ship-in-the-bottle action.

And this is a lot of fun for us because not only do you have the self-folding constraint, but you have this enclosure constraint. These forms would not look as exciting if they could sprawl out, and they didn't have any constraints to live inside these bubbles. So you get yet another collection of forms from that glass.

All right. I think that's the end of our little sculpture tour. We'll come back to sculpture at the end of this lecture. But I want to talk more about-- oh, question.

AUDIENCE: What was the idea behind the blindfold?

PROFESSOR: What was the idea behind the blindfold? So paper folding is all about touching material. Glassblowing is usually very visual. It's about you look at the material, and you don't usually touch it. So we wanted to unify the two in order to put them together. Plus it was just a crazy idea.

I think what happened is I was in my office. My dad calls me up. He's blowing glass. He's like, I got this great idea. Come over with a camera. So in that video, I was the cameraman. And I was like, all right, you want to burn yourself blowing glass. He didn't actually burn himself. Usually, he does. But blindfold, he was more careful. Yeah, blindfold glassblowing, it's pretty crazy. That was his first try ever blindfold glassblowing in that video.

All right. So this hyperbolic paraboloid, it's been around for a long time. I didn't mention the history. It goes back to the Bauhaus in the late '20s. Albers, who I'm sure many of you know of, taught a class about design.

And he liked using paper as a material that would force you to focus-- not worry about material in the sort of architectural scale and what would stand up-- and just think about design. And paper folding was really tactile and good for that.

And I'm not sure exactly whether it was him or a student of his, it's somebody in that period, 1927, '28, came up with this model and the circular one. And then it's been taught many times since then. Then it really hit it big in the origami community in the 1980s by [INAUDIBLE], and since then everybody's been folding it. Question?

AUDIENCE: Is it not rigid, but paper [INAUDIBLE]?

PROFESSOR: Ah, you made one. Can I show it? Thanks, [INAUDIBLE]. Simple little example, but here is concentric squares, and you crease it down. Normally you get this kind of X. You let go, and it pops into a little saddle. So this is a sort of low-resolution one. If you spend twice as much time, you can double the resolution and so on.

Your question, is it rigid? Probably not rigid. There's like this degree of freedom.

AUDIENCE: Rigid in the sense of that between creases [INAUDIBLE].

PROFESSOR: Yeah. What's happening in between the creases?

AUDIENCE: Exactly.

PROFESSOR: That is exactly the topic of today's lecture. In fact, we have a paper called "How Does Paper Fold Between Creases" to address exactly this question. I'm going to hang onto it, if you don't mind because it'll be useful to point at.

In fact, it is completely impossible to fold that crease pattern into this shape. And so your idea that there's something weird going on is an idea-- we've known for a long time that these faces could not stay planar. I mean, at least visually it's hard to see unless you stare at it with all the right angles. But the faces look like they twist.

Now, that's OK. Paper can do all sorts of curving stuff without creasing. I mean, this is sort of like a twist, no creases involved. So we thought this was possible, but in fact, it's not. So this is the big surprise. It's something we discovered just last year, and I'm going to prove that to you today.

So the theorem is, if I have this concentric square crease pattern, even ignoring the mountains and valleys, it is impossible to fold that crease pattern into anything that is not flat. So you can, of course, not fold it at all, then none of the creases get folded. You can collapse it all the way down. That probably isn't even allowed. But never mind.

But what we really want is a 3D form where every crease is bent by a non-zero angle and also not by 180 degrees. So we'll call that a proper folding, something where every crease is strictly between 0 and 180, which is what we want in these 3D forms. And the theorem is, that is impossible for the hyperbolic paraboloid. There is none.

So that's weird because we fold them all the time. We've been folding them for like 11 years, and other people have been folding them for 80 years. What's happening with the real piece of paper? Well, one possible answer is that there's more creases that you don't see in this model. And if you add more creases, it is possible to fold something that looks like a hyperbolic paraboloid.

So here we have the regular crease pattern in black, and then I've added some purple diagonals. Wherever we had a black trapezoid, I've added one purple diagonal to triangulate the crease pattern. And here I've chosen what seems to be an especially good triangulation where I zigzag back and forth within one quarter. And then also from around a ring, I zigzag back and forth.

So that triangulation folds into this. That's another theorem. This is construction on a computer, obviously, where we have 16 rings, and the central crease here is folded by an angle of 30 degrees. That's the theta. Notice I also had to remove part of a diagonal. That's also necessary. In the center, you can't crease both of those, the folds in an X, by a non-zero amount.

Those are all things we're going to prove today. But before we prove the negative things, I want to prove the positive thing, that this thing can actually be built. That may seem obvious because here it is. I built one. But because there's all these arguments of, well, here it is, I built one, we've got to be especially careful about, does this thing really truly exist?

So let me tell you how it's constructed and how we can actually prove not only can we build it approximately on a computer, but we can prove there really is one there even though we don't have it exactly. So it's actually pretty cool and pretty easy.

So this is the triangulated hyperbolic paraboloid. So the idea is to work from the inside out. We're going to start with this central square here-- and let me draw it over here-- fold it by some angle theta. That, I think, we all know how to do. It's some rotation matrix. And then I want to work my way out.

So at this point, I know the location of these four vertices. In general, I know some square and everything inside. I want to figure out everything in the next square. And the creases are going to look like this, from my zigzag pattern.

So there's stuff in here, which is known. So I already have figured out the location of these vertices. And I want to know, how do I figure out these vertices on the outside? If I can do that, I just repeat, and I get a bigger and bigger hyperbolic paraboloid.

So what I do is pretty easy. I'm going to look at this vertex first and say, well, I have these three points, and I have known distances between those three points. I'm going to assume here, and we'll see why later, that each of these creases remains a straight line in three dimensions. It's not obvious, because creases might bend around.

But let's assume that we were going to fold it in a simple way. These triangles are going to remain triangles. These edges will remain straight. And they have to remain the same length because we're paper folding. You can't stretch. And so I know these three points in 3D, and I know these three distances, just measuring them on the crease pattern, flat crease pattern.

So this point is on the intersection of three spheres, and the centers of the spheres are not collinear. This is a trick we used last class but in two dimensions. So I have three non-collinear points. I have three spheres-- little hard to draw-- centered at them.

The intersection of two of the spheres is going to be a circle, and the intersection with the third sphere is going to be actually two points. So the intersection of three spheres is two points. So a little bit of ambiguity.

But it turns out one of these points will set the mountain-valley assignment correctly, and the other one will set it incorrectly. Like if I want this to be a mountain, one of them will be inside and one of them will be outside. One of them will make this mountain, one will make it valley.

So it turns out if you actually do this, it's uniquely determined at every step of the way. Because you know what mountain-valley assignment you're aiming for, you uniquely figure out what this point is. By the same reasoning, you can figure out what this point is, intersect these three spheres.

And then once I know those two points, I can figure out this point because it's the intersection of these three spheres, with now these points are known. And then I can figure out this point in the symmetric way. OK? That's how you do it. We repeat. And that is exactly how this model is built.

Now, this is still not a proof that it really folds. It's a construction method. I'd say it's an algorithm. If I start out with the inner thing here and I give you some angle, I can approximate where these vertices are. And then I can keep going and, at each step of the way, approximate where the vertices are.

The worry is the approximation gets worse and worse because I have this propagating error effect. The larger n is, the number of rings in my hyperbolic paraboloid, the lower the accuracy will be. But I get to choose, at each step of the way, how much precision I use to compute all of these numbers.

Let me tell you how this computation actually works. I did this in Mathematica, or we did this in Mathematica. Here's Mathematica. And I asked Mathematica, well, here is the equations for the intersection of three spheres. I say, well, squared Euclidean distance between this point and some unknown thing x, y, z is this distance squared. And here's the second one, and here's the third one. It's kind of tiny.

And then here's the answer. It's a little messy. In fact, you can ask how many terms are in this thing. It's 444,000 terms in the solution. Don't try this at home, I guess, by hand. Intersection of three spheres is a bit messy. There's probably a cleaner way than what Mathematica does, but this is an easy way to get it in.

Now, in theory-- so if you look at all of this stuff, all that's happening is you have all the various inputs, x1, y1, x2, z2, and so on. You have some numbers like 4 and minus and 2. All you're doing is taking these numbers, adding them, multiplying them, dividing them, squaring them, subtracting them.

And at some point, taking square roots. I don't know if we'll ever see that. I mean, if we search through, there are some square roots, but they're going to be in very specific places. Yeah, so I probably won't find one instantly. Oh, there's one, square root.

So all I'm claiming at this point is you can compute the intersection of three spheres just using basic arithmetic and square roots. This is called a radical expression, not because it's so amazing and controversial. But it's radical because that symbol is called rad, I guess, and radical refers to square roots. I guess they were crazy at the time.

All right. Actually, I want to stay with Mathematica. So that's the idea. Still not a proof, just a construction method. But here's a trick, which you can use to turn a construction into an actual proof that this thing exists. It's called interval arithmetic, little computer science lesson.

How many people have heard of interval arithmetic? No one-- one person, all right. Yeah, even computer scientists probably don't necessarily know this unless they've seen some numerical analysis.

So it's an idea that instead of computing an approximate location for this point, I want to get not only an approximate location, but also an error bound on how much I don't know it. So there are three coordinates to every point. For each coordinate, every number I want to represent I'm going to represent as an interval from some lower bound to some upper bound.

So say, well, I don't know what the value is, but I know that it is somewhere in between these two numbers. So represent every number like this. And then you just need to define, how do I add numbers, subtract them, multiply them, divide them, and take square roots? And the answer is carefully.

I'll show you one example. If I have two numbers, L1, U1, and L2, U2, and I want to add them together, then I believe it's pretty simple. It's just L1 plus L2, and U1 plus U2. This is in perfect mathematical world where there's no round off.

In reality, when you add two numbers, you lose a bit of precision. And you make sure that when you add these numbers, you always round down, and when you add these numbers you always round up. So that way, these intervals, the accumulation error is realized by these intervals getting wider. You can start with them super tiny, maybe, in fact, 0 length because you know exactly where these points are, so they're 0 intervals.

But then, because of the error in every operation you do on the computer, the intervals will widen. But as long as you do this computation with enough bits of precision, they'll widen slowly. If you do it slowly enough-- if you do it with enough precision, and the errors accumulate slowly enough, you can build your n-ring hyperbolic paraboloid without any error, or without too much error to make things go wrong.

What could go wrong is that these three spheres might not intersect. That's really the worry here. There are two things that could go wrong. One is that the spheres don't intersect. The other is that the surface intersects itself. Both would be bad.

How do you tell when the spheres don't intersect? Well, we have this formula, this ginormous formula. The only thing that could go wrong is that you take a square root of a negative number. Now, hopefully that never happens in reality.

But what could happen for us is we have such a poor approximation of our numbers. And we try to take the square root of some interval, and L is less than 0. Now, probably the actual number is more than 0, but we can't tell, and we just know the number is somewhere in this interval. If the lower bound is negative, then we can't take the square root. We don't know what the right answer would be.

So Mathematica conveniently can do all of this for you. If you just plug in intervals instead of numbers, it will do interval arithmetic correctly, and you can tell it what precision to do. And that is how we found this example. In fact, I will show you the notebook, and there's all this computation and stuff. This is all, I believe, it's in our paper, so if you want the code. So we compute the crease patterns so I can measure all the distances. And there's some collision detection and stuff to make sure everything's working. And it computes some rings.

And then, actually, the one I want to show you is this one. This is model we've been looking at, and here it is in three dimensions. The way it's being rendered is a bit odd, but you get the idea. That is the hyperbolic paraboloid we construct.

Now, in reality, each of these points is actually a little interval. It's hard to draw that because it's smaller than the resolution of the screen. I think I compute up here the- - this is like how much we've messed up the edge lengths. It's like all of these 9's, and then this little area at the end. OK. Cool.

So the point is, you just do this with enough precision. As long as you don't end up computing any negative square roots, you get a surface where every point is actually a little box. You don't know where the point is exactly in that box.

Then you check for collision, just knowing that the points are somewhere in those boxes. You try to intersect two triangles. As long as they intersect away from the boxes, you're OK. Then you know there's no actual intersection.

And we have done that for this triangulation up to-- get it right-- up to n equals 100, 100 rings, and where the theta angle here is any even number between 2 and 178 degrees, so 2 degrees, 4 degrees, up to 178 degrees. We're not interested in 180 because that would be flat. That's not interesting. We're not interested in 0. Every even number of degrees in between. Why even? Just because of the representation.

I'm sure it works for the odd. In fact, I'm pretty sure it should work for any theta and any n. But this technique will only let us to prove it for specific theta and specific n. Because we're just using a computer, it's only going to check one example. We don't have a nice way to do them all at once.

In case you're interested-- oh, here I've built it out of sheet aluminum, I think?

AUDIENCE: Galvanized--

PROFESSOR: Galvanized steel? All right. Water jet cut along the creases. Now, steel's a little tricky to not add extra pieces by accident, so there's a few defects. But this is also some kind of verification that it works.

In the computer here, the number of digits of precision. So for whatever reason, Mathematica speaks base 10 instead of base 2. Which is weird to me, but maybe intuitive to everyone else, non-computer scientist, I guess. So there's normal sort of floating points like 16 digits of precision or so. And there you can build this thing up to n equals 3, 3 rings.

But you go up to about 1,000 digits of precision, and then we can get beyond 100. It depends, though, on what the angle is that you fold. So for angles that are very big, close to 90, we had to go up to 2,000 digits of precision. And I don't know how far n can go here. I think maybe a couple 100, but I'm not sure exactly. The larger n is, of course, the more accumulation you have, and so you have to do every operation with more digits of precision.

But the conjecture would be, any specific data and n, there's some precision for which this approach will work, but we don't actually know how to prove that. All right. So that's a triangulated hypar.

Here's another triangulation, which we also studied in this paper, where instead of zigzagging up one quarter of the square, we just do all the diagonals in the same direction. We still zigzag around a ring but not between rings. And this can also fold, as the other model that was in the Mathematica file. I won't bother going there.

This is for a small fold angle of just 8 degrees in here, n equals 16. And there's a reason I did it for a small angle, because it doesn't work for large angles. So that triangulation we started with was a good one. It's actually not the first one we tried, but it is sort of natural.

If you fold by a very small amount, you can get all the way up to 133 rings, but then it fails after that. The spheres just don't intersect anymore. Depending on how much you fold, like if I fold by 22 degrees, I can get up to 13 rings.

But here I wanted to get to 16 rings so I only went up to, what was it, 8 degrees or something. I could have gotten a little beyond but not a lot. And up at 178, you can still get three rings, but it doesn't fold after that. So triangulation matters, which triangulation you do.

Going back to the one that works well, natural question is, is it actually a hyperbolic paraboloid? It's called a hyperbolic paraboloid because-- or there is a surface called a hyperbolic paraboloid, where if you take a cross-section this way, you get a parabola. That's the paraboloid part.

And if you take a cross-section this way, you get a hyperbola. That's a little harder to draw, but it goes around here and around here. Hyperbolas have two connecting components.

So let's look at the parabolic part here. We're supposed to be approximating a parabola if we look at all the points on the top here and also maybe on the bottom. So that's what I've drawn. The green lines, which are very light so a little tricky to see, they zigzag back and forth, and that is what we compute. Again, it's approximate, but it's so close to accurate that I just draw them as single points. The error is much smaller than the thickness of these lines.

And then the blue and purple lines here are fits of the best parabola that matches these points. Actually, it's not even the best parabola. It's kind of funny. I just take the last three points-- three points determine a parabola. I take the parabola that fits through those three points and bam! It is almost a perfect fit.

These are the error charts, I guess. This is relative errors, probably the most informative. Or actually, it's the ratio-- it's not really an error-- the fit value over the actual value.

So when it's 1, that means it's perfect. And yeah, this is at the end of the chain where I fit it, so of course it's going to be perfect out here. At the center, at the very beginning-- so I'm only looking at one quarter of this thing-- yeah, the error is a little bit.

The ratio is not 1, but it's 0.9997 or so, maybe a little bit less, so that I had to write it down. It's like 0.003% error, if I got it right, maybe just 0.03% error. Fix the notes. Something like that. It's very small is the point. But it is non-zero.

I mean, initially we thought, well, maybe if you increase the resolution-- you make a finer and finer hyperbolic paraboloid-- it will more closely approximate a hyperbolic paraboloid. That does not seem to be true because, really, making it finer is really just like making it bigger. You never really change the center behavior. It's always the same.

And this construction proves it. We build the center, and we can go as far out as we want. It's just changing the scale of the thing. But the center will always remain the same, and it will always remain off the parabola, but super, super close. Really, it's all right to call this a hyperbolic paraboloid, but you should triangulate, especially if you're making something out of more rigid material. Cool.

Let's go on.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, question.

AUDIENCE: Do the creases have to be uniform? I mean, could you tighten up inside?

PROFESSOR: The creases do not have to be uniform. Most of the hyperbolic paraboloids we've made, we do evenly space all the squares, but they don't have to be. I say that in that we've made them out of paper, and they fold to something, and you get other kinds of surfaces. Probably not going to be a hyperbolic paraboloid anymore.

I have not done it with Mathematica and checked that it really is possible, but it should be. That would be a fun thing to explore at some point. Other questions?

All right. So this is the end of the positive news for hyperbolic paraboloids. Now we're going to go to the negative stuff, showing that it is impossible to fold this with this crease pattern. Hyperbolic paraboloids don't exist without triangulation.

For this we need a little bit of math tools in this paper, "How Paper Folds Between Creases." So in the study that we just did, I assumed that all of the creases stayed straight. And therefore that all of the faces-- therefore? Yeah, I guess, because triangles are rigid. If I forced the edges of the triangle to stay straight, then the interior of the triangle must stay flat just to preserve distances.

So I assumed that every face of the piece of paper stayed flat. Why did I assume that? Because paper, real paper, does not have to stay flat in between the creases. This is real stuff here. But paper is constrained on how it can curve without creases, and that is the purpose of this little mathematical endeavour.

There's some technical stuff, and I want to get to the meat as quickly as possible. But I'll just mention some assumptions. We assume that the thing that we fold, the folding of our piece of paper, is piecewise C2. C2 means you can take two derivatives, and it's still good and continuous and sort of smooth up to the second level.

Piecewise means, of course, we have creases. Those are not C2, not even C1. So we have some crease pattern on our piece of paper. It's whatever. And what we mean is that inside one of these regions, it's C2. On the creases, it's nothing. It's C0. It's continuous. We don't rip the paper. But we don't necessarily have derivatives everywhere. So that's the assumption.

Now, one annoying thing here is you can have something called a semi-crease, something we call a semi-crease, which is C1 but not C2. Yeah. So you have to divide into pieces, but it's not technically a crease. And this is some of the worry of what might be happening.

Maybe you don't need creases here, you only need semi-creases. Maybe you only need to violate the second derivative, not the first. Creases should be violation of first derivative. Those are sharp things. Semi-creases are just kind of little sharp.

It's like-- what's a good example of a semi-sharp thing? I guess if I take a parabola, and then I take a more shallow parabola, then at this point there's no second derivative because the first derivatives don't meet. This one has a first derivative of this. This one has a first derivative of that. They're not the same. Is that true? Parabolas maybe not. But you get the idea. This is a discontinuity in the second derivative, not the first.

All right. I'm going to basically ignore semi-creases here, though, because they're just kind of a technicality. They don't end up mattering. So we have piecewise C2. And we assume that our surface is intrinsically flat, meaning it came from a piece of paper-- trinsic-- oh, boy, intrinsically flat.

Now this is something we know as curvature zero. And I want to tell you a little bit more about curvature. We have defined curvature in a few different-- well, I guess in one particular way, which is you add up the material, and you take 360 minus that. So curvature zero means you have 360 of material, which is good news.

But there are other ways to define it, in particular, the way Gauss defined it. This is usually called Gaussian curvature because he invented it. Gauss defined curvature of a 3D surface. You have some weird curved 3D thing, like this shape, the orange part. And there's actually a lot of natural notions of curvature here. Gaussian curvature is just one of them.

So let me tell you a little bit about that. We're looking at this point, and I want to compute the curvature of this point. Well, if I look in any direction, like say this red direction-- so it's like I slice this world with a plane, this blue plane, I get a nice one-dimensional curve.

Then the curvature is how bent that curve is at that point. That's like a directional curvature. In this direction, how bent is-- what does how bent mean? It just means you try to nestle a circle in there, in that plane, and you take 1 over the radius. That's a curvature. OK? So if it's flat, the radius could be infinite, and so curvature zero. If it's very sharp, then the radius is very small. And so 1 over the radius is very large, big curvature.

So it's some thing here. Let's say it's a positive number. And then if I take, for example, this other blue plane, so this cross-section, in that direction the curvature is bent the other way. So you say, well, there the directional curvature is negative. It depends which way is up. One of them is positive, one of them is negative. Somewhere in between it's going to be 0. But you have all these different directional curvatures.

The Gaussian curvature, which is the one that we sort of know and love and have used a lot, mostly for polyhedra, is the min directional curvature times the max directional curvature. Now, I really mean min and max. So the smallest one, in this case, is negative. The largest one is positive. You take the product, and that's the Gaussian curvature. It's a weird thing.

But in particular, because there's a negative one and a positive one here, that means the product is negative, and that's because this is a saddle. Still the case, negative Gaussian curvature means saddle. Positive means a convex cone. 0

means intrinsically flat.

So this is still the thing we know, but this is a weird way of thinking about it. This is how Gauss defined it. And he proved that even if you fold the surface, the Gaussian curvature never changes. That is called the theorema egregium. Cool name.

So Gaussian curvature doesn't change under folding. We start with something that's flat, zero curvature. Because if you take a flat plane, you take any directional curvature, everything's 0. So it's the product of min and mix. They're both 0. So we start with something zero curvature, it will remain so.

Now, if I have a product of two things, and I know this is equal to 0, that means one of the two things is 0, maybe both. If it's still a plane, both of them will be 0. But one of them still has to be 0. What this means is, basically, locally at any point, we have a cylinder, some kind of generalized cylinder.

But it really only curves in one direction because there's some direction where it doesn't curve at all. Everything's straight. In all the other directions, yeah, it curves different amounts. The orthogonal direction will be where it curves the most. That will be the max, I guess, and the min will be 0.

So that is what a folded sheet looks like. And it's maybe not so obvious, but when I did all this contorting and what not, really I was only bending in one dimension, like a cylinder. That's not exactly a cylinder because I can change the radius of the cylinder all over the place, but I'm really only bending in one direction.

I was reading on Wikipedia, this explains how we eat pizza. Because you take a piece of pizza, which is basically like a piece of paper, and if you bend it a little bit, like you push in the center and push up on the sides, you give it some curvature in one direction. And therefore it has to remain straight in the other, so it kind of supports the piece of pizza. Never thought of it that way, but there you go, practical applications for this stuff.

All right. These things we call planar points. If I have a point p and it's locally flat like a plane, call it planar. These things, for some crazy reason, we call parabolic. This is to be consistent with other notation. I know it looks-- cylindrical would be another fine term. But there's a third kind called a elliptic, which would be like this stuff. That doesn't happen.

So we just worry about locally cylinder, which we're going to call parabolic because it's also like locally a parabola, doesn't matter, and locally planar. Those are the two kinds of points we can have.

Now, I'm going to give you some accelerated facts we're just going to take as given from differential geometry. Well, really, we had to take some facts that were about differential geometry and sort of port them to our context. They didn't give us exactly what we wanted. We had to generalize them a little. Differential geometry, if you were at Sunday's lectures, that was one of the main tools being used there.

Differential geometry is about smooth things, like C4 usually. Now, sometimes about C2 things. It is almost never about piecewise C2 things. So you got to worry about the pieces. Now, most the time, we're thinking about one little region here, and that's nice in C2. And you can check. Most of the differential geometry still applies there.

Blah, blah, blah. Let me tell you some facts. If I take a smooth point, so that means not on a crease and not a semi-crease, then it lies on something called a rule segment, which is a line segment, also called a rule line. And the endpoints of that segment, here's the interesting part, are on creases or the boundary.

So we already know from this picture that any point lies on a segment, like an actual 3D line segment, because of this parabolic nature. What's interesting is that segment, it can't stop. Just keeps on going, like the Energizer bunny, until it hits a crease. At that point, things aren't smooth. We don't know what happens. But really, these creases go straight. Those are rule lines. And here, actually, we have a choice, many different rule lines.

But in the parabolic case, that rule line is unique. It's going to be unique for parabolic points. And so we get what's called a ruled surface, which is just the union of a whole bunch of line segments, rule lines, around any point. Maybe any smooth point, just to be safe.

So you may have heard of ruled surfaces. They're quite common. They're fun because you can build them out of strings. So each string, if you hold taut, it's a line segment. Take a whole bunch of them and just imagine the envelope there. That is a 3D surface, and that's a ruled surface. Now this one does not have zero curvature. It has negative curvature everywhere. So that can't happen.

This actually can happen. This is like you take a helix, and so there's a blue curve in 3D, a space curve. And then you imagine taking the tangent at every point. So just like if you just went straight at every point instead of turning, then you get a bunch of lines. Those are rule lines. And you get this cool surface.

Now, that is a valid folding of a piece of paper. Doesn't look like a cylinder, does it? But locally, each of these lines looks like a cylinder. It's just the radius of the cylinder is changing all over the place. But that's a valid folding of a piece of paper, I think. Has zero curvature everywhere, if I did it right. Or I didn't do it, but if I imagine it correctly.

All right. So it's a ruled surface, great. I mean, I can really create my whole surface locally around a point by a whole bunch of rule segments. They're not all going to be parallel or anything, which is what we imagined from the cylinder. They can turn around.

But these segments do keep going until they hit another crease. Maybe the blue line's a crease. And the boundary here could be the boundary of the paper, could be a crease boundary. But that's what they look like.

Now, it's also what we call torsal, torsal ruled. Here we get to somewhat more obscure terminology but some useful things. It's quite restricted. I want to have a common tangent plane throughout a rule segment. So if I look at any one of these rule segments, there's one plane, which is going to be like this, that is tangent to the surface at every point along that segment.

So in general, you might imagine that the tangent plane turns as we go along. But these segments are kind of-- they're locally cylindrical. So you can really make a tangent plane all the way along. Whereas here, that's probably not true. Imagine the tangent plane starts-- can you see my hand? It starts to bend like this, and it's going to bend around like that. It twists along a single line.

In our case where we have zero curvature, it's actually going to be torsal ruled, meaning the tangent plane just goes straight along each of these lines. We're going to need this, that's why I mention it.

Another fun fact along the same spirit is that the points along a rule line are all the same in terms of whether they are planar or parabolic. So I'll call them uniformly planar slash parabolic. Remember, planar just means it lies in a plane. Parabolic is the other case. So here everything's parabolic. But you can't like suddenly switch in the middle and become flat. That's not allowed, not possible.

So those are some fun facts. We're not going to prove them. If you want to see the proofs, they're a bit technical. You have to read our paper, and then the differential geometry books we cite. But a bunch of these things, you can understand the proofs just assuming a little bit of differential geometry, and it's not too hard. It's in our paper, but it's a little bit technical.

So I want to look at the things that are more related to paper folding, how paper folds. This is obviously related, but it's like the foundation on which we build what I care about. What I care about most-- all right, let me tell you a fun fact, what we're going to prove.

We want this nice proper folding of the hyperbolic paraboloid, meaning every crease is bent by a non-zero angle and not 180 degrees. In that situation, first claim is every crease remains straight. Second claim is every face remains rigid, can't bend anywhere-- every interior face.

These guys on the boundary, they can do crazy things. And you see that in the model, where the outside gets kind of all wiggly. That's allowed. The outside faces, the ones that share the boundary with the paper, can wiggle, can curve. But everything else has to remain flat if there are no creases in there. And that's not at all obvious. So we're going to prove it. And I need some water.

All right. First claim, I'll call it polygonal implies flat. So what I mean is, suppose I have some region of paper. It's in the middle somewhere. Let's say it's smooth. And I look at the boundary of the region. Doesn't actually have to be smooth. It could have semi-creases. But it has no creases inside.

Suppose the boundary in 3D is a polygon, so it's piecewise straight. I don't know how that corresponds to anything, but say the boundary is straight. Then the inside must be planar. So if I have a polygonal boundary, then I have a flat planar inside. This is assuming no creases.

OK? So what that means is that every point inside this region must be planar, not parabolic. That's what we want to prove. So let's do a proof by contradiction, and suppose that we have a parabolic point. So we have some point, and locally it is curved, and we know there's a rule line through it.

I know by smoothness, by continuity, that if this guy is parabolic, it's bent, then, in fact, all the points nearby should also be bent, because you can't go instantly from bent to straight. You've got to do that slowly. So there's some little region-- we call this a neighborhood-- around the point that is all parabolic. So I'll call this parabolic neighborhood.

All right. All of those points have unique rule lines. That's what we've been saying. So I take this little neighborhood, and each one of them defines some rule line. Those rule lines go all the way out to the boundary something like this.

Now, this boundary is the boundary. That means it's polygonal, can't be curved like this. In fact, it looks something like this. Maybe this is straight. Maybe this has two segments. We don't know how many segments. I just want to look at one of these segments here and all the rule lines coming out of it.

So what I have is I have the boundary of the paper. That's a poor imitation of a straight line. Straight line. I have some rule lines coming out. Now, I don't really know what they look like. I want to understand what they look like. This is in 3D. Imagine. OK, this is straight.

First thing I want to look at is the normal to the surface. So normal is like perpendicular to the surface. It's easy to define normals in the interior of the surface, but I can actually extend that out to the boundary.

So I want to look at a normal here. Maybe let me put it here, something like that. If I took normals really close to the boundary, I just took the limit out to the boundary, I'll get some normal vector. And by smoothness, that exists. Just wave my hands there, but that's true.

OK, what's true about this normal? Well, it's perpendicular to the surface. Now locally, the surface here, it's defined by the plane of this line segment and this line segment. So in fact, it's perpendicular to the boundary, and it's perpendicular here to the rule line. Buy that?

If I take some other normal, like this one-- sorry, like that, it's also perpendicular to this, but it's perpendicular to some other rule line now. So these guys are perpendicular to a common line, but they may not be the same direction.

What we do know, though, is that it's torsal. There is a single tangent plane that's perpendicular to this entire rule line, which means the normal is the thing perpendicular to that tangent plane. So in fact, all of the normals along this line are identical, which is kind of neat. These guys are all parallel. Same thing here. I won't draw that. It's going to get too messy.

So what? All right. Well, here's a crazy thing to imagine. I will look at the derivative of the normal. So I have this point p. Imagine you have a point p, and it's moving along this curve continuously. Sorry-- not curve. This is a straight line. It's moving along this edge of the boundary just once. It goes like this.

And I'm going to define n of p is the normal at that point. So it starts at something, and then maybe it's changing. I get to this normal, then I get to this normal, and then who knows what it looks like. I want to understand, can it change at all? So to understand its change, I'm going to take the derivative, n prime of p. So this is as p moves along here, how does the normal change?

I claim, in fact, it can't change at all. Why? Well, the derivative of the normal, first of all, must be perpendicular to-- I haven't given anything a name here. This thing is the boundary edge. So here's my boundary edge. Every single normal here-- I didn't draw the greatest picture-- of those have to be perpendicular to this one common edge.

Here's the fact where we use that this is a straight line. If it curves, then this is not a consistent thing. But because it's straight for a while, all of these guys are perpendicular to the same thing. All of the ends are perpendicular to the boundary edge. This is just little perpendicular notation.

And if all of the n's are perpendicular to the boundary edge, the change in n must also be perpendicular. Otherwise, it would change in such in a way it's longer perpendicular. OK? So that's just sort of intuitive. Great.

What else? I need it to be perpendicular to something else. I claim that n prime of p is also perpendicular to the rule line. Ah, yes. Because this normal-- all of these points along a rule line have the same normal direction. These guys were all parallel. And so if I look at the change in n, in order to make all these guys be the same, I also cannot change n in such a way that it has a non-trivial-- I'm not going to say this too well. Do you believe it, more or less?

Let me try to say it once. I'm looking at the change in the normal. I don't want the normal to change in such a way that it will change along this axis. So for that to be true, the change in the normal must be perpendicular to this direction.

This is kind of weird because the normal is also perpendicular to this direction. I mean, the normal satisfies the same things. It's perpendicular to the boundary edge. It's also perpendicular to the rule line. That means that n, the normal, and its derivative, they have the same direction.

That's weird. In fact, it means that the derivative must be 0. Because the normal is always unit length. So this would be saying that the normal is getting bigger or shorter, but it can't do either. So in fact, normal's not changing at all. So in fact, all of these guys are parallel along the whole segment. That means this whole region is flat, and that's a contradiction. All right? So I waved my hands a little bit, especially on this step, but believe me.

All right. That's one fun fact, but we want more. Because how do we know that the boundary would be polygonal? For that we need that straight creases remain straight. And something's not true when you have curve creases. Curve creases obviously don't remain curved. But curve creases don't even remain planar. I mean, they can do crazy things. But straight creases stay straight.

So what I mean is, I have my flat piece of paper. I take a straight line in the piece of paper, and I say that's going to be a crease. It has no first derivative along the crease. Then when I fold that, I get a straight line and some crazy stuff on the other side, but that crease remains straight.

So I was in the middle of the paper, so it actually looks like this. Remains a straight segment, obviously of the same length. But the point is it can't curl, and it can't even kink.

Now, just any good theorem needs a counter-example. So if we take a piece of paper and we make a straight crease, look, I can curl that crease. So this theorem is not true unless I say that the crease is proper.

So here I had to make the crease all the way to 180 degrees, then I can curl it. Also, if I don't fold it at all, then I can curl it. But if it's folded something in the middle, I cannot curl. Paper will not be happy with me. I need extra creases in order to curl. Well, yeah, it's just not possible.

So proper crease, which means the fold angle is not equal to 0 or plus or minus 180. So that's the theorem. Let's prove it. Maybe go a little faster. All right. So we have our crease, and in three dimensions it might look curved. We take some point.

We know that the paper is flat.

So if you look at the left side and the right side of the piece of paper, locally, at least, it's flat. And I want to think of there being some tangent plane on the right side and some tangent plane on the left side. Wow, that's not a good picture.

Let me draw the planes first. So here's two planes, and then here's my point p. And locally the surface actually lies in these two planes and kinks here. The surface kinks. We want the crease to be straight here, but maybe the crease is bent something like that.

All right. Here's what we do. Again, I want to think of-- in this case, I'm going to look at tangents, not normals. So I'm going to say, well, every point on the curve here has some tangent. In fact, the tangent at this point must lie right along the intersection of these two planes.

I'm going to give these planes names. This is p. This is going to be tangent plane Tp, and this is going to be other tangent plane Sp, so on the left and the right side. So the normal at p-- sorry, not the normal, the tangent at p-- did I give it a name-- p prime. That's what you'd normally call it. That's the derivative. Tangent of p lies along the intersection of those two planes. We don't really need that, but it's true. Give you some intuition.

Now, what I'm really going to use is the second derivative, little more extreme here. Second derivative is curvature on the curve. Now, we know that in the unfolded surface, there's exactly 180 degrees of material here. Here's some point p. We know there's 180 degrees of material. I mean, locally, everything's flat in terms of the surface.

So here's where I'm going to wave my hands a little bit. I claim that the curvature vector must be perpendicular to Tp and perpendicular to Sp. Because if I take this curve, and I project it into one of these planes, say Tp, it should be straight at p in projection.

It also must be straight when I project into Sp. And that actually means that it's straight in three dimensions also. So the straightness and projection corresponds to straightness when flattened basically. I'm waving my hands a little bit.

But if I need the curvature vector to be perpendicular to this plane-- so it's something like this. It's coming straight out of this plane. But that uniquely determines its direction. It also needs to be perpendicular to Sp. These two vectors can't be the same thing, and yet they have to be, unless these planes are the same.

So it could be there's a zero fold angle, then it's fine. Or it could be it's 180-degree fold angle, then again the planes are the same. Can go in one direction or the other. But if I have a proper crease, then there's no such vector. And so we're done. That proves straight creases stay straight. Cool. Any questions about that?

So I'm trying to hit a middle ground of not too technical but also not too shallow. And so you're left in the middle state of sort of make sense. But I did wave my hands, and it's not super rigorous. Because to be rigorous it would take forever, and I need to teach you differential geometry and so on.

But now that we are armed with these tools-- straight creases stay straight and any polygonal boundary stays flat, planar-- that tells us that if we have some crease pattern, any crease pattern made out of straight edges, so no curve creases allowed, if I make straight creases, every single interior face must stay rigid in any proper folding.

That's pretty cool. Because we've talked about rigid origami and said not much is known about rigid origami. It's tough. But in fact, you really need to understand rigid origami if you want to fold something like a hyperbolic paraboloid. Now, when you do flat folding stuff-- well, I guess those are kind of rigid also.

But any kind of 3-dimensional folding, where you fold every crease somewhat but not all the way, all the faces need to be rigid in the final form. Now, you could get there by all sorts of means. But we want the hyperbolic paraboloid to exist with creases only along these lines. And now we know each of these trapezoids must remain planar. Also, each of these triangles must remain planar.

Now, now we get to the contradictions. If you have this piece of paper and those are your only creases, you can fold along one of the creases or the other, but not both. Unless you fold one crease all the way to 180-- here, we can do it.

I have one diagonal, and I have another diagonal. So everything's meeting at 180 degrees. I can fold one and then the other. I can fold one, or I can fold the other. But I cannot fold a little bit of one, and then try to fold the other too. It's not possible. Therefore, this crease pattern doesn't fold at all. OK? Kind of trivial. The center is messed up.

But keep in mind, again, these faces we don't know anything about because this edge could do crazy things. But the interior faces of the crease pattern must all stay flat and stay exactly as they were in the original. So it's really like we have rigid panels here and hinges between them. The outside, though, we have no idea what happens. OK. So the inside sucks.

So I think, at this point, we get to this picture. Here's the regular crease pattern. So let's just make life easier and say, well, we'll cut out a hole in the center. That surely-- that must be better. But it's no better. That thing still can't fold because you take any of the rings that goes around, like a square annulus-- here's a ring.

The creases in that ring are just those four. So I'm just looking at these four trapezoids. Again, each one must stay planar. And these are the only hinges I have. It's actually really just like this. I've cut a hole in the center, but it won't really matter because the extension of these lines meet at a point.

So effectively, you are folding this whole triangle. You can fold just the trapezoid. But however you fold the trapezoid, you could just extend it and make the triangle, and these triangles will remain meeting. So it's actually the same thing as the center diagram. Because this doesn't fold, this also won't fold.

It doesn't matter how many layers you cut out. As long as there's one interior layer that doesn't touch the boundary, it's going to be screwed. OK. That is why hyperbolic paraboloid doesn't fold without diagonals. Once you know the creases are straight, it's kind of obvious.

But let me give you a more general picture. This makes things a little more interesting and tells you, really, all those straight crease pleatings that I've shown you, like hexagons and octagons, none of them are possible, which was a surprise.

So here I'm taking concentric shapes in such a way that the diagonals meet at a point. I still need that fact. If they don't, in fact, it's possible to fold. And we have a little example of that.

OK, now I actually need two interior rings to get a contradiction. So I take some ring like this, and I take another ring, and I suppose none of them touch the boundary. As long as I know there's at least four, there's going to be two consecutive rings like this. And the hinges are the diagonals and the inner ring boundaries.

OK. If I just look locally at this, I claim it must be mountain-mountain-mountain-valley or the reverse. This is something we would normally prove by saying, OK, this is locally smallest, therefore these guys are different. And this is also locally smallest, therefore these guys are different, and so on. Except that only works for flat folding. Here I need to prove it for 3D folding. I'm not going to bother, but you can do it.

So in fact, let's say this is mountain-mountain-mountain-valley, then this must be mountain, this must, all the way around must be mountain. OK? That's not such a big deal. This must all be valley, and this must all be mountain, or the entire reverse. But the point is, all of these guys out here must be mountains.

So really if I extend all of these guys and they meet at a single point, if I was lucky, that's what I'm supposed to assume, really what I'm doing is folding a whole bunch of triangles where all of the creases are mountains. That's not possible.

So if you didn't believe this thing, it follows from the same argument. They can't all be mountains, and therefore some crease is not getting folded. So it's the same argument again, you just have to be a little more precise. And that is not folding hyperbolic paraboloids and things like that. But if you had diagonals, it all works.

Now, in the last few minutes, I want to show you some more fun things, in particular, pillows. Everybody likes pillows. But how are they made? Well, it's like two squares of material joined along the edges, and then you stuff stuff inside. That kind of weird, because doubly-covered square I think of as flat, and yet you can put material inside.

What is the maximum volume you can stuff into a pillow? This is called the tea bag problem because it also works for square tea bags. Open. We don't know. Lots of practical explorations, but open.

Another version of this is Mylar balloons. Mylar balloons can't stretch, more or less. And here is two circles glued together. Again flat, but you can pump air into them. You get this weird behavior, these ripples on the outside. That's real stuff. But we don't know what the maximum volume shape is, but it should look something like what we see in reality.

Let me tell you some fun theorems. If you take any convex polyhedron, like say a cube, you can inflate it and increase its volume. Also, I guess, works for a doubly-covered square. This is the tea bag open problem, where these edges are joined.

How do you do it? I'm going to quote from the paper-- "by simultaneously delivering karate chops to the edges of the polyhedron." That's in the abstract. So you take an edge like this-- here's an edge and the two incident faces-- and I go like this.

Wow, that was exciting. So what happens when you go like this-- the whole board shakes, I'll probably break it eventually-- you get something like that. So there's a valley here, and then mountains there to replace this mountain.

And you can prove, if you do that with suitable parameters here, at every edge simultaneously you increase the volume, which is pretty neat. Now, how far can you go? What happens when you keep increasing the volume? Is the limit polyhedral, or is it smooth with ripples?

Conjecture is it's smooth with ripples. In fact, you can prove it's not polyhedral. Because not only for convex polyhedral, you take any polyhedron, it is possible to inflate it by at least some volume. So the limit has to be curvy. It's not going to be polyhedral. Now, exactly what it looks like we don't know, but lots of experiments to that effect.

A big open question in this field is, does this exist? We think so. It seems curve creases-- none of this stuff works for curve creases. Curve creases seem a lot more powerful. We haven't been able to prove this exists because it seems very flexible. There's a lot of degrees of freedom. It's hard to figure out where the rule lines are. But we think so.

So what about curved creases? I haven't talked about curve creases really at all until this lecture. I would say the most seminal work in curve creases was done by David Huffman. Huffman is super famous. He was a grad student and a professor here at MIT for many years.

In particular, when he was a grad student he came up with these Huffman codes, which are used in every MP3 player. Every device you use that uses any kind of compression has Huffman codes in it. Super cool from the '60s.

But he also did a lot of curve crease origami, curve crease folding. And over the last couple of years, we've been working with his family, so his wife and two daughters, Linda and Elise. And this is me, that's Marty, and this is Duks Koschitz, who many of you know is a PhD student in architecture. And we were visiting there back in May in particular.

And what we're doing is taking his work, which is all in their houses and almost no one has seen, and figuring out how he made them. What is the underlying mathematics? What are the crease patterns that make it possible? And then recreating them to check that we did it right.

So what I have here are just a few examples of our recreations so far. So these are not the original models, but they look just like the original models, made out of the same material and more or less the same way.

Although we do it a little more high-tech than he did because we want to draw perfect computational diagrams in CAD, and then reproduce them exactly on the paper with no reproduction errors. So we use fancy robotically-controlled devices to do that. So there's some fun curve creases. You get some really nice shadow patterns. All of those are circular arcs, all the creases.

Here are some more. These are actually parabolic arcs and some straight creases in between. Get some cool 3D relief effect. These are like tessellations like Tom Hull was talking about but with curve creases.

This is pretty awesome. Here the creases are quite complicated to figure out. But what's happening is you're taking a cone like this and pleating it back and forth but with different angles, and so the whole thing twists. We have a physical one of these in our offices. But this is the 3D model of what's going on.

Here's some particularly awesome crease patterns. David Huffman made a whole variety of these. These are just a couple of examples. Got some crazy circular arcs. You've got some-- I guess these are also circular arcs and some straight segments.

And then you wrap this around to make a cylinder, and you get this. That one-- there's a bunch of photos of the original David Huffman model on the web if you Google around. Not looking exactly like that, but there's a whole variety.

Here's another one, a bit of a maybe surprise what it looks like. Most of the center just gets eaten away from these creases. Here the creases are elliptical, and you get something like that. So there's some hidden structure beneath.

I think just one more example. This is probably the most coolest puzzle. You say, well, you float along these parabolas, just one valley and one mountain. But you fold them a lot, you get that. And Huffman composed these to make various kinds of tubes and really cool things.

So this is work in progress. Just wanted to give you a sample. There's a paper about this which shows more examples. But our goal is to recreate all or most of them and document how he did it and figure out how he designed these models. He had mathematical tools to do this, and we're still figuring out what they are. The hope is it will lead to more great mathematics and art about curve creases.

And that is it. This is my last lecture. Next class we have-- or next three classes we have a whole bunch of student presentations. Please come. There will be lots of awesome things there. Send me your slides ahead of time, and you'll have 10 minutes to do it, to give a talk.

And then the last lecture, Wednesday two weeks from now, is Tomohiro Tachi. And thanks very much. It's been a lot of fun doing this class and having you all as students.

[APPLAUSE]