Improve SNR (Signal to Noise Ratio)

$\rightarrow$ use large areas (harder to compute, sensitive to edge orientation)

w $\boxed{-1}\ \boxed{+1}$

Another view: $M = \sqrt{E_x^2 + E_y^2}$ $\quad \frac{dM}{ds} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial s}$

$$\frac{dM}{ds} = \frac{E_x E_{xx} + E_y E_{xy}}{\sqrt{E_x^2 + E_y^2}} \cdot \frac{E_x}{\sqrt{E_x^2 + E_y^2}} + \frac{E_x E_{xy} + E_y E_{yy}}{\sqrt{\dots}} \cdot \frac{E_y}{\sqrt{\dots}} = \frac{E_x^2 E_{xx} + 2 E_x E_y E_{xy} + E_y^2 E_{yy}}{E_x^2 + E_y^2}$$

$$\frac{dM}{ds} = 0 \Rightarrow \begin{vmatrix} E_x & E_y \end{vmatrix} H \begin{vmatrix} E_x \\ E_y \end{vmatrix} = 0. \quad (1)$$

Example: $E(x,y) = f(r)$ where f is such as: 

Q: does the curvature change where the edge is detected?

$E_x = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} = f' \frac{x}{\sqrt{x^2+y^2}}$ $\quad E_y = f' \frac{y}{\sqrt{x^2+y^2}}$ $\quad E_{xx} = \frac{\partial^2 f}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial r}\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}}\right)$

$E_x^2 + E_y^2 = (f')^2$ ... we plug-in in (1) and: $\begin{vmatrix} E_x & E_y \end{vmatrix} H \begin{vmatrix} E_x \\ E_y \end{vmatrix} = \frac{f'^2 f''}{f'^2} = f''$

$\rightarrow$ the answer is: it's independent.

Estimating Derivatives: $E_x^2 + E_y^2 = \left(\frac{E_{01} - E_{00} + E_{11} - E_{10}}{2\varepsilon}\right)^2 + \left(\frac{E_{10} - E_{00} + E_{11} - E_{01}}{2\varepsilon}\right)^2$

$E_x^2 + E_y^2 = \frac{1}{4\varepsilon^2}\left[(E_{00} - E_{11})^2 + (E_{01} - E_{10})^2\right]$ $\quad \frac{1}{2\varepsilon}\begin{array}{|c|c|}\hline & +1 \\ \hline -1 & \\ \hline\end{array} \quad \frac{1}{2\varepsilon}\begin{array}{|c|c|}\hline +1 & \\ \hline & -1 \\ \hline\end{array}$

→ 1965 Robert's gradient operator

Equivalent cascade of smaller operators

$y = (x \otimes f_1) \otimes f_2 = x \otimes (f_1 \otimes f_2)$

ex: $\frac{1}{\varepsilon}\begin{array}{|c|c|}\hline -1 & +1 \\ \hline\end{array} \otimes \frac{1}{\varepsilon}\begin{array}{|c|c|}\hline +1 & +1 \\ \hline\end{array} = \frac{1}{2\varepsilon}\begin{array}{|c|c|c|}\hline -1 & 0 & +1 \\ \hline\end{array}$

$\frac{1}{\varepsilon}\begin{array}{|c|c|}\hline -1 & +1 \\ \hline\end{array} \otimes \frac{1}{\varepsilon}\begin{array}{|c|}\hline +1 \\ \hline +1 \\ \hline\end{array} = \frac{1}{2\varepsilon}\begin{array}{|c|c|}\hline -1 & +1 \\ \hline -1 & +1 \\ \hline\end{array}$

Q: given a big filter, can we decompose it into smaller ones? NO!

Huge speed-up in 2D: basically $N^2 \to 2N$.

Separably rot. symm. filter

$f(x) f(y) = g(r)$ (does not depend on θ). $\frac{\partial}{\partial\theta}(f(x)f(y)) = 0$

$f'(x)\frac{\partial x}{\partial\theta} f(y) + f(x) f'(y) \frac{\partial y}{\partial\theta} = 0 \Rightarrow f'(x) - r\sin\theta f(y) + f'(y) f(x) r\cos\theta = 0$

$\Rightarrow y f'(x) f(y) = x f(x) f'(y)$

Separate variables: $\frac{1}{x}\frac{f'(x)}{f(x)} = \frac{1}{y}\frac{f'(y)}{f(y)} = K$ $\quad f(x) = e^{kx^2/2} \quad f(x)f(y) = e^{\frac{k}{2}(x^2+y^2)}$

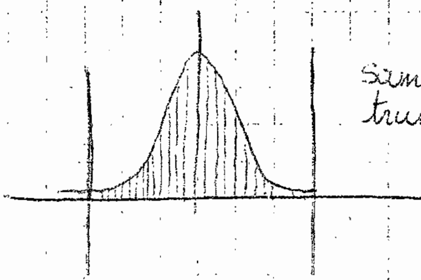
Now we want $f \otimes y \ldots \quad h(x) \otimes h(y) = i(r)$

$h(x) = \text{IFT}(f(\omega)) \quad h(y) = \text{IFT}(f(\omega)) \quad \text{IFT}\left(e^{-\frac{1}{2}x^2/\sigma^2}\right) \longleftrightarrow e^{-\frac{1}{2}\sigma^2\omega^2}$

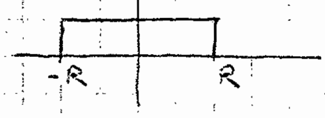
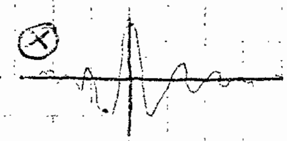
$e^{\frac{-1}{2}\frac{x^2}{\sigma^2}}\delta(y) \otimes e^{\frac{-1}{2}\frac{y^2}{\sigma^2}}\delta(x)$

"one-d convolution"

Discrete Equivalent

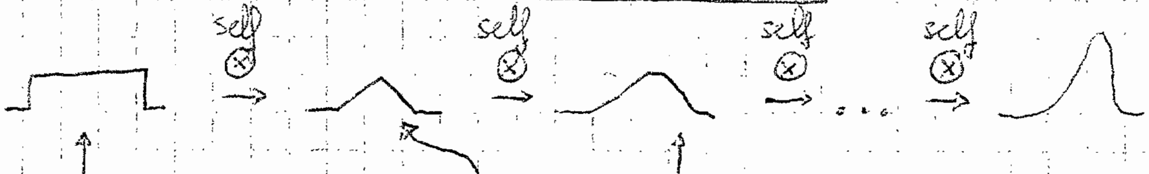


simple truncate

Truncation:  $\Longleftrightarrow$ $\otimes$ 

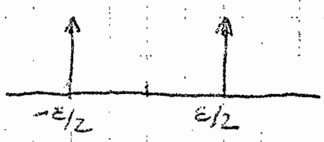
It is not an appropriate discrete equivalent
→ lost nice property on high-frequency

$\boxed{\text{Central Limit Theorem}}$



gaussian

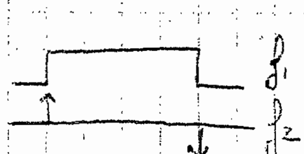
$\frac{1}{2}\,|+1|+1| \otimes \frac{1}{2}\,|+1|+1| = \frac{1}{4}\,|1|2|1| \ldots \frac{1}{8}\,|1|3|3|1| \to \ldots \to |(x+y)^n|$ binomial filters



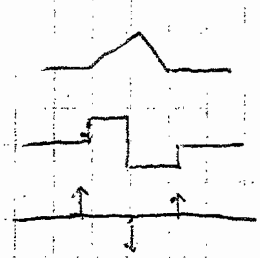
$$\int_{-\infty}^{\infty} \left(\frac{1}{2}\,\delta(x+\frac{\varepsilon}{2}) + \frac{1}{2}\,\delta(x-\frac{\varepsilon}{2})\right) e^{-j\omega x}\,dx = \cos\frac{\omega\varepsilon}{2}$$

Convolution $\Longleftrightarrow$ multiplying by $\cos\frac{\omega\varepsilon}{2}$ $\to$ $\boxed{\cos^n\frac{\omega\varepsilon}{2}}$ smoothly dies at HF ≥ 0

<u>Speed-ups</u>

Block Averaging 

$F_{i_0} = \sum_{j=i_0}^{i_0+m} E_i \qquad F_i = F_{i-1} + E_{i+m} - E_{i-1}$ can be extended to piecewise constant

Next step: piecewise linear 

Next step: piecewise quad.