**Handed out:   2004 Dec. 2nd**                    **Due on:   2004 Dec. 9th**

**Problem 1:**    Suppose we allow for a scale factor in the problem of absolute orientation—in addition to translation and rotation—so that the transformation from the "left" to the "right" coordinate systems becomes

$$\mathbf{r}_r = sR(\mathbf{r}_l) + \mathbf{r}_0$$

Then the absolute orientation problem becomes one of minimizing

$$\sum_{i=1}^{n} \left\| \mathbf{r}_{r,i} - sR(\mathbf{r}_{l,i}) - \mathbf{r}_0 \right\|^2$$

(a) By differentiating w.r.t. to $\mathbf{r}_0$, dividing by $n$, and setting the result equal to zero, find a formula for the best fit translation involving the centroids $\bar{\mathbf{r}}_l$ and $\bar{\mathbf{r}}_r$ of the two sets of measurements

(b) By shifting the origin of each set of measurements to its centroid, show that the translation $\mathbf{r}_0$ drops out and the error to be minimized simplifies to

$$\sum_{i=1}^{n} \left\| \mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i}) \right\|^2$$

where $\mathbf{r}'_{l,i} = \mathbf{r}_{l,i} - \bar{\mathbf{r}}_l$ and $\mathbf{r}'_{r,i} = \mathbf{r}_{r,i} - \bar{\mathbf{r}}_r$.

(c) Show that this can be written in the form

$$S_r - 2sD_{rl} + s^2 S_l$$

where $S_r$ and $S_l$ are sums of squares of the lengths $\mathbf{r}'_{r,i}$ and $\mathbf{r}'_{l,i}$ respectively, while $D_{rl}$ is the sum of dot-products of $\mathbf{r}'_{r,i}$ and $R(\mathbf{r}'_{l,i})$.

(c) Conclude that the best fit scale is given by

$$s = D_{rl}/S_l$$

(d) Now suppose we instead find the best fit transformation

$$\mathbf{r}_l = s'R'(\mathbf{r}_r) + \mathbf{r}'_0$$

from "right" to "left" coordinate system. We might expect that this transformation is the inverse of the earlier one, that is, $s' = 1/s$, $R' = R^{-1}$, and $\mathbf{r}'_0 = (1/s)R^{-1}(\mathbf{r}_0)$. Unfortunately this is not so. In particular show that

$$s' = D_{lr}/S_r$$

where $D_{lr}$ is the sum of dot-products of $\mathbf{r}'_{l,i}$ and $R'(\mathbf{r}'_{r,i})$. In general $s' \neq 1/s$.

(e) Show that this asymmetry can be removed by instead minimizing

$$\sum_{i=1}^{n}\left\|\frac{1}{\sqrt{s}}\mathbf{r}'_{r,i} - \sqrt{s}\,R(\mathbf{r}'_{l,i})\right\|^{2}$$

What is the best fit value for $s$ in terms of $S_r$, $S_l$, $D_{rl}$ and $D_{lr}$?

(f) Show that the best scale factor as defined in part (e) can be determined without knowing the rotation or translation.

(g) Does the best fit rotation depend on which of the three choices for best fit scale factor ((c), (d) or (e)) we adopt?

**Problem 2:**
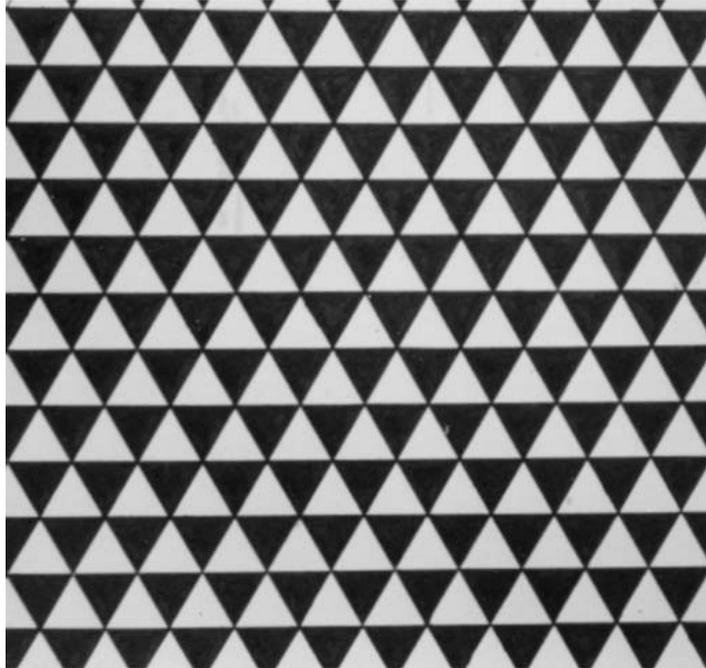


The above is a photograph of a portion of a flat target used for camera calibration as well as estimation of camera distortions.

(a) Explain image processing methods that could be used to accurately recover the images of the vertices of the pattern.

(b) Explain the advantages—if any—of this pattern over one consisting of three sets of parallel black lines along the edges of the triangles in this pattern (on a white background).

(c) Explain the advantages—if any—of this pattern over one consisting of sets of black dots at the vertices of the pattern (on a white background).

(d) Explain the advantages—if any—of this pattern over one consisting of LED light sources placed at the vertices of the pattern (on a black background).

**Problem 3:**   Representing finite rotations using unit quaternions.

(a)  Consider a tetrahedron with vertices at $(a, 0, b)^T$, $(a, 0, -b)^T$, $(-a, b, 0)^T$, and $(-a, -b, 0)^T$. What should be the ratio between $a$ and $b$ so that this is a regular tetrahedron? Find the unit quaternions representing all rotations that bring the tetrahedron into alignment with itself.

(b)  If we start with the tetrahedron in a different alignment with the coordinate axes we get a different set of unit quaternions representing the group of rotations of the tetrahedron. Show that if $\{\mathring{q}_i\}$ is the set of quaternions found in part (a), then the set of quaternions found with a different alignment of axes can be written in the form $\{\mathring{p}\mathring{q}_i\mathring{p}*\}$ for some unit quaternion $\mathring{p}$

(c)  One measure of the "size" of a rotation is simply the angle of rotation. If we ignore the identity operation, what is the "smallest" rotation amongst the groups of rotations of the regular polyhedra (Platonic solids)?

(d)  Show that
$$(\mathring{a}\mathring{q}) \cdot \mathring{b} = \mathring{a} \cdot (\mathring{b}\mathring{q}*)$$

Show that
$$(\mathring{a}\mathring{b}) \cdot (\mathring{a}\mathring{b}) = (\mathring{a} \cdot \mathring{a})(\mathring{b} \cdot \mathring{b})$$

Show that if $\mathring{r} = (0, \mathbf{r})$, $\mathring{s} = (0, \mathbf{s})$ are quaternions representing vectors then
$$\mathring{r}\mathring{s} = (-\mathbf{r} \cdot \mathbf{s}, \mathbf{r} \times \mathbf{s})$$

Show that if $\mathring{r}$, $\mathring{s}$ and $\mathring{t}$ are quaternions representing vectors then
$$(\mathring{r}\mathring{s}) \cdot \mathring{t} = [\mathring{r}\,\mathring{s}\,\mathring{t}]$$

(e)  Show that the formula for rotating a vector $\mathbf{r}$
$$\mathbf{r}' = (q^2 - \mathbf{q} \cdot \mathbf{q})\mathbf{r} + 2q(\mathbf{q} \times \mathbf{r}) + 2(\mathbf{q} \cdot \mathbf{r})\mathbf{q}$$

can also be written in the form
$$\mathbf{r}' = (q^2 + \mathbf{q} \cdot \mathbf{q})\mathbf{r} + 2q(\mathbf{q} \times \mathbf{r}) + 2\mathbf{q} \times (\mathbf{q} \times \mathbf{r})$$

and that if $\mathring{q}$ is a unit quaternion, then the latter formula actually requires fewer arithemetic operations to evaluate than the former.

**Problem 4a:**   Least Squares Image Adjustment.

In the classical least squares approach to photogrammetry, one minimizes the sum of squares of differences
$$\sum_{i=1}^{N}(x_{Ii} - x_{Pi})^2 + (y_{Ii} - y_{Pi})^2$$

between observed image positions $(x_I, y_I)$ and predicted image positions $(x_P, y_P)$ based on scene coordinates and camera parameters. Perspective projection gives us
$$\frac{x_{Pi} - x_o}{f} = \frac{x_C}{z_C} \qquad \text{and} \qquad \frac{y_{Pi} - y_o}{f} = \frac{y_C}{z_C}$$

This approach is used whether one is trying to recover the parameters of the imaging situation (interior orientation, exterior orientation, and so on), or recover coordinates of points in the environment.

Given image measurements of a feature point $(x_{I1}, y_{I1})$ determine the best fit coordinates of the corresponding point $(x_{C1}, y_{C1}, z_{C1})$ in the scene by minimizing the sum of squares of errors. Comment on the result. What happens when $N > 1$?

**Problem 4b:** In the case of (i) absolute orientation, (ii) exterior orientation, and (iii) interior orientation, one can use two-dimensional versions of the three-dimensional problems in order to gain some insight. We didn't do this for relative orientation.

In the case of linear 'cameras' operating in the plane, what is the minimum number of correspondences of rays from the 'left' camera with rays from the 'right' camera that are needed to fully constrain the relative position and orientation of the right camera with respect to the left camera?

**Problem 5:** This problem address a possible ambiguity in interpreting motion fields.

By differentiating the perspective projection equation

$$\frac{1}{f}\mathbf{r} = \frac{\mathbf{R}}{\mathbf{R} \cdot \hat{\mathbf{z}}}$$

with respect to time $t$, we get an equation for the motion field $\dot{\mathbf{r}} = d\mathbf{r}/dt$. Next, in rigid body motion we have for the velocity of a point in space relative to the camera:

$$\dot{\mathbf{R}} = -\mathbf{t} - \boldsymbol{\omega} \times \mathbf{R}$$

where $\mathbf{t}$ is the instantaneous translational velocity of the camera, while $\boldsymbol{\omega}$ is the instantaneous rotational velocity.

Now consider a camera looking at a planar surface. Suppose $\mathbf{R}_0$ is an arbitrary point in the surface. Lines connecting points in the surface, such as $(\mathbf{R} - \mathbf{R}_0)$, are all perpendicular to the surface normal $\mathbf{n}$ of the plane. That is,

$$(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{n} = 0$$

Show that the motion field when

$$\mathbf{t} = \mathbf{a}, \quad \mathbf{n} = \mathbf{b}, \quad \text{and} \quad \boldsymbol{\omega} = \mathbf{c}$$

is the same as when

$$\mathbf{t} = \mathbf{b}, \quad \mathbf{n} = \mathbf{a}, \quad \text{and} \quad \boldsymbol{\omega} = \mathbf{c} + k\,\mathbf{a} \times \mathbf{b}$$

for suitable choice of the constant $k$. What is the value of $k$?

Draw a diagram showing an example of this kind of ambiguity where two different motions and two different surfaces give rise to the same motion field. (Note: In order to make things simpler, you may want to pick $\mathbf{c} = 0$.)