Elasticity

(and other useful things to know)

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*** With thanks to Steve Senturia, from whose lecture notes some of these materials are adapted.**

Outline

> **Overview**

> **Some definitions**
- **Stress**
- **Strain**

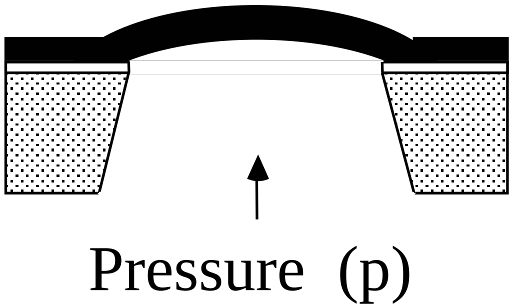
> **Isotropic materials**
- **Constitutive equations of linear elasticity**
- **Plane stress**
- **Thin films: residual and thermal stress**

> **A few important things**
- **Storing elastic energy**
- **Linear elasticity in anisotropic materials**
- **Behavior at large strains**

> **Using this to find the stiffness of structures**

Why we care about mechanics

> **Mechanics makes up half of the M's in MEMS!**



Pressure sensors

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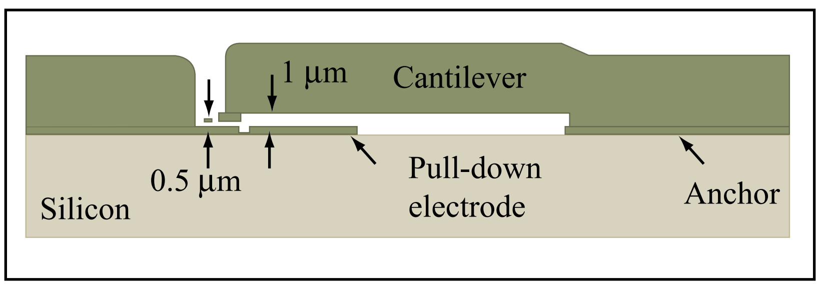

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Adapted from Rebeiz, Gabriel M. *RF MEMS: Theory, Design, and Technology*. Hoboken, NJ: John Wiley, 2003. ISBN: 9780471201694.

Switches ***Zavracky et al., Int. J. RF Microwave CAE*, 9:338, 1999, via Rebeiz *RF MEMS***

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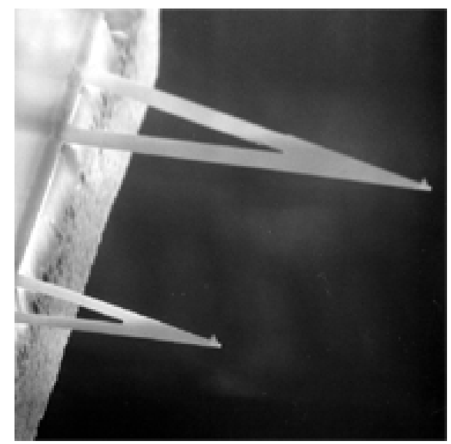

Veeco.com

AFM cantilevers

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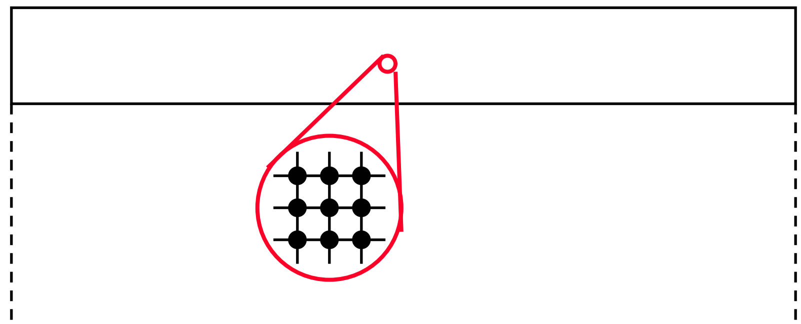
What do we need to calculate?

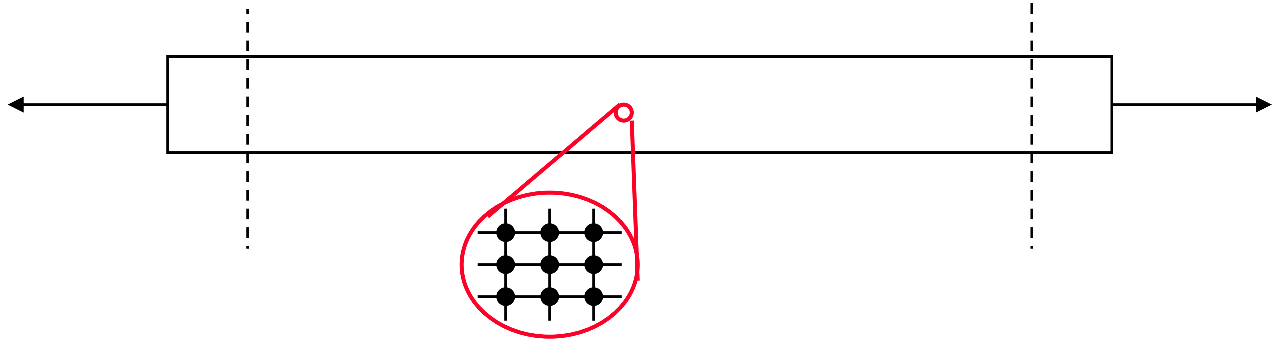
- **Eager beaver suggestion: everything**
  - **When I apply forces to this structure, it bends.**
  - **Here's the function that describes its deformed shape at every point on the structure when the deformations are small.**
  - **Here are numerical calculations of the shape at every point on the structure when the deformations are large.**
  - **The structure is stressed, and the stress at every point in the structure is…**
- **Shortcut suggestion: just what we really need to know**
  - **When I apply a force F to the structure, how far does the point of interest (the end, the middle, etc) move?**
  - **This boils down to a stiffness, as in F = kx**
  - **What is the stress at a particular point of interest (like where my sensors are, or at the point of maximum stress)?**
  - **How much load can I apply without breaking the structure?**

Why things have stiffness I

Unloaded beam is undeformed:



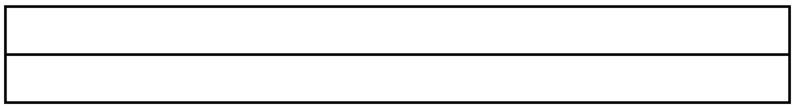
Axially loaded beam is stretched:



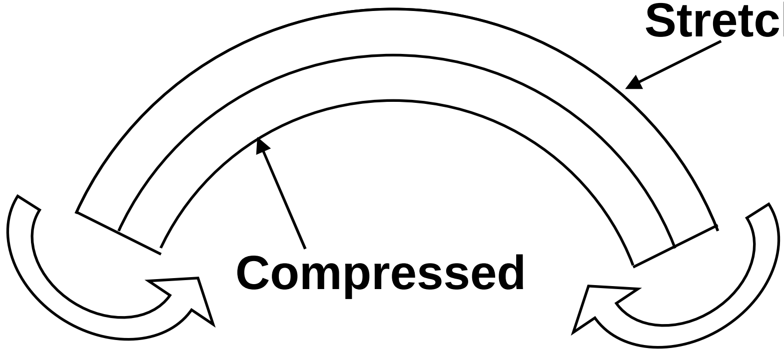
Stretching costs energy, which is stored as elastic energy. Exactly how much energy is determined by material and geometry.

Why things have stiffness II

Unloaded beam is undeformed:



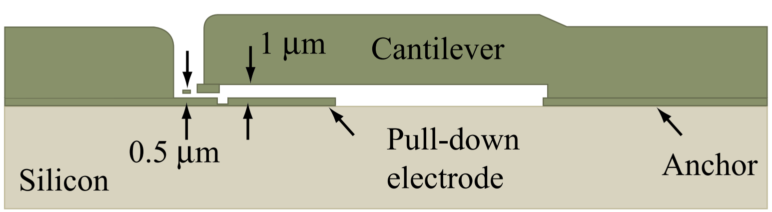
Loaded beam is bent:

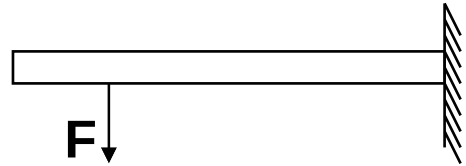


Stretching and compressing cost energy, which is stored in elastic energy. Exactly how much energy is determined by material and geometry.

Example: relating load to displacement in bending

> What are the loads, and where on the structure are they applied?





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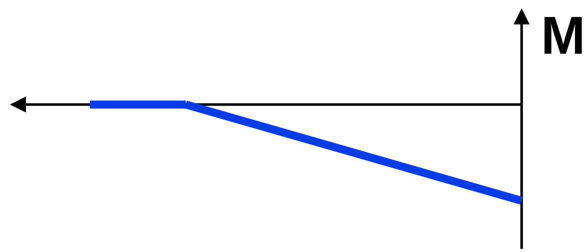
> Given the loads, what is going on at point (x,y,z)?



> How much curvature does that bending moment create in the structure at a given point?

- What is the geometry of the structure?
- What is it made of, and how does the material respond to the kind of load in question?

Elasticity

- **Elasticity: the ability of a body to deform in response to applied forces, and to recover its original shape when the forces are removed**
- **Contrast with plasticity, which describes permanent deformation under load**
- **Elasticity is described in terms of differential volume elements to which distributed forces are applied**
- **Of course, all real structural elements have finite dimensions**
- **We will ultimately use partial differential equations to relate applied loads and deformations**

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- Isotropic materials
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  - Plane stress
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Stress

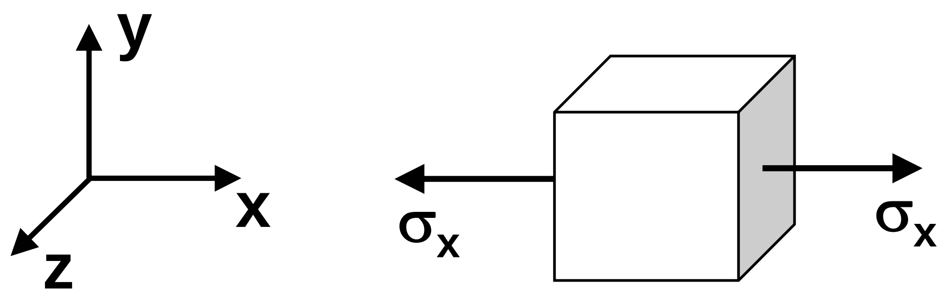
- Stress is force per unit area

- Normal stress

  σ_x, σ_y, or σ_z

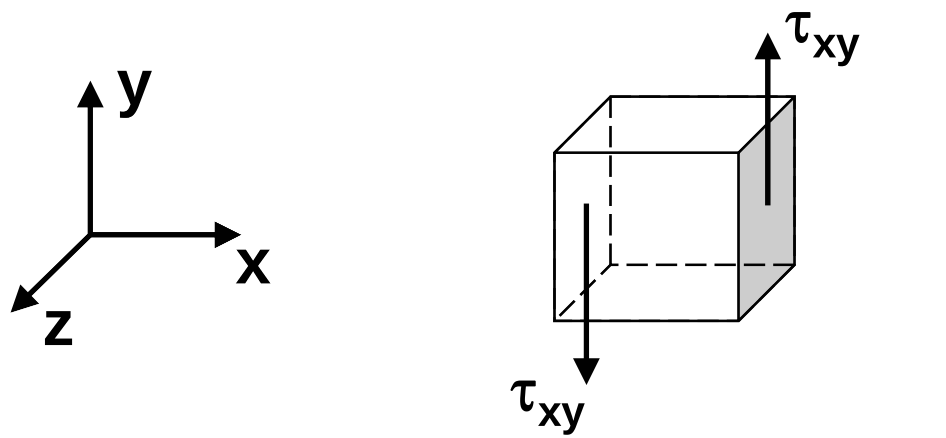
- Compressive: $\sigma < 0$

- Tensile: $\sigma > 0$



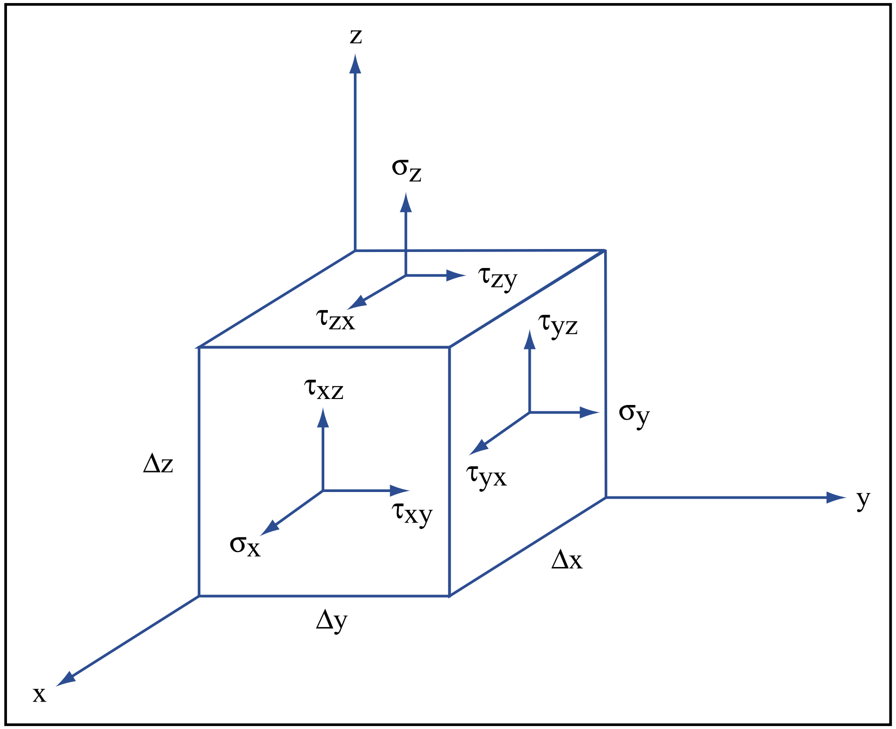
- Shear stress

  τ_{xy}, τ_{xz}, or τ_{yz}



Stress

- Can have all components at a given point in space
- SI Units: the Pascal
  - 1 Pascal = 1 N/m^2
- Other units:
  - 1 atm = 14 psi = 100 kPa
  - 1 $dyne/cm^2$ = 0.1 Pa
- Notation: $\tau_{face,direction}$



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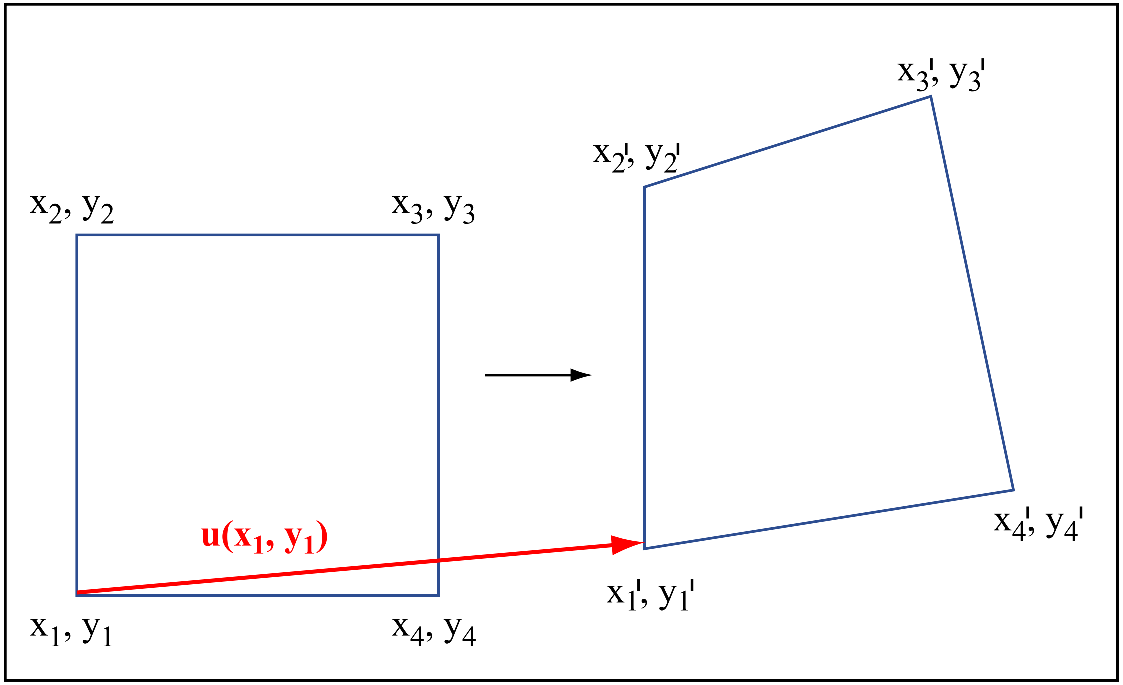
Deformation

> **Illustrating a combination of translation, rotation, and deformation**



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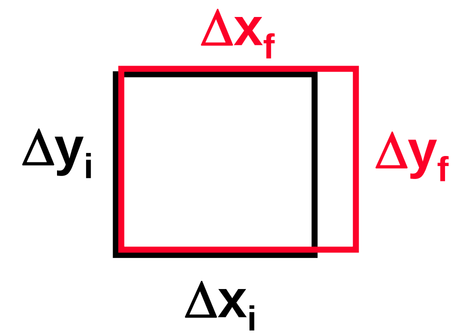
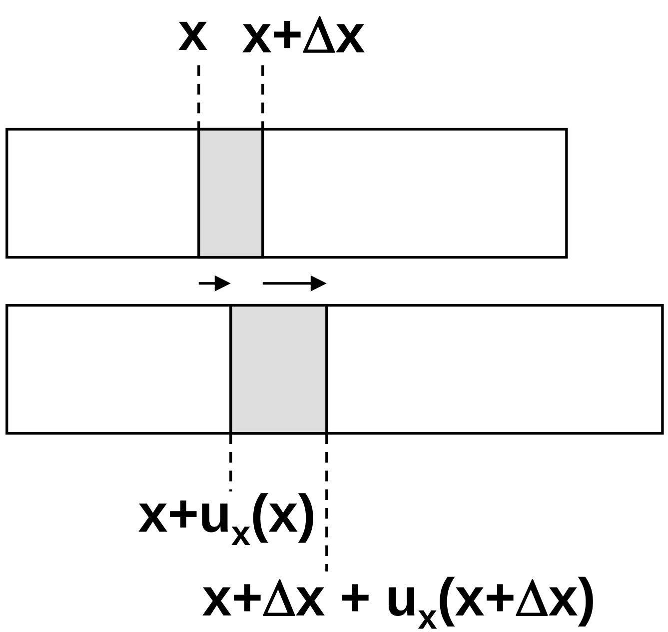
Strain

- Strain is a continuum description of deformation.
- Center of mass translation and rigid rotation are NOT strains
- Strain is expressed in terms of the displacements of each point in a differential volume, u(x) where u is the displacement and x is the original coordinate
- Deformation is present only when certain derivatives of these displacements u are nonzero

Normal Strains (ε_x, ε_y, ε_z)

- Something changes length
- Normal strain is fractional change in length (dimensionless)
- $\varepsilon > 0$: gets longer, $\varepsilon < 0$: gets shorter





$$\text{Initial length}: (x+\Delta x) - x = \Delta x$$

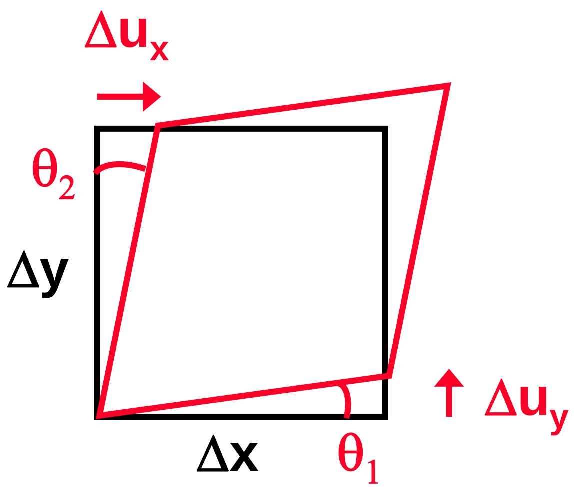
$$\text{Final length}: \big(x+\Delta x + u_x(x+\Delta x)\big) - \big(x + u_x(x)\big) =$$

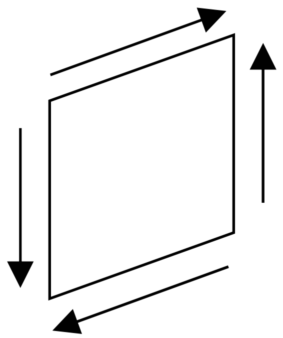
$$= \Delta x + u_x(x+\Delta x) - u_x(x)$$

$$\varepsilon_x = \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} = \frac{\partial u_x}{\partial x}$$

Shear Strains (γ_{xy}, γ_{xz} , γ_{yz})

- **Angles change**
- **Comes from shear stresses**
- **Quantified as change in angle in radians**





$$\gamma_{xy} = \left(\frac{\Delta u_x}{\Delta y} + \frac{\Delta u_y}{\Delta x} \right) = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

$\approx \theta_2$ $\approx \theta_1$

Different regimes

> **How are stress and strain related? It depends on the regime in which you're operating.**

> **Linear vs nonlinear**
> - **Linear: strain is proportional to stress**
> - **Most things start out linear**

> **Elastic vs. plastic**
> - **Elastic: deformation is recovered when the load is removed**
> - **Plastic: some deformation remains when unloaded**

> **Isotropic vs. anisotropic**
> - **Life is simpler when properties are the same in all directions; however, anisotropic silicon is a part of life**

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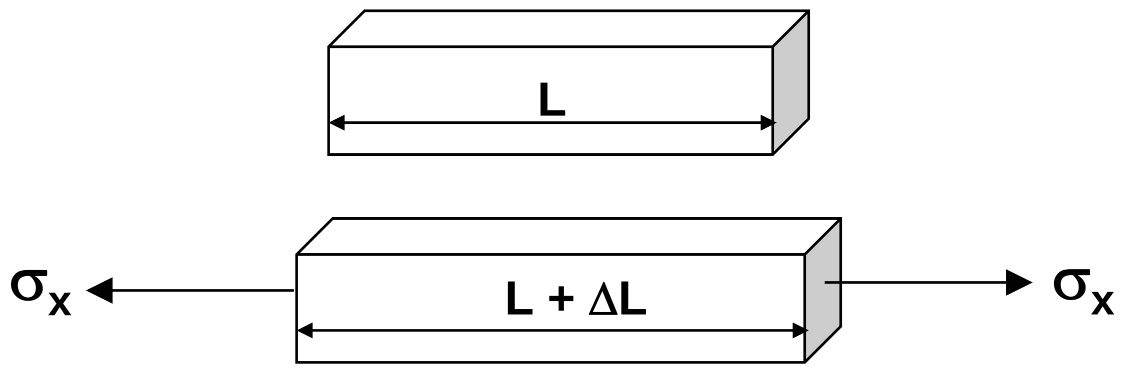
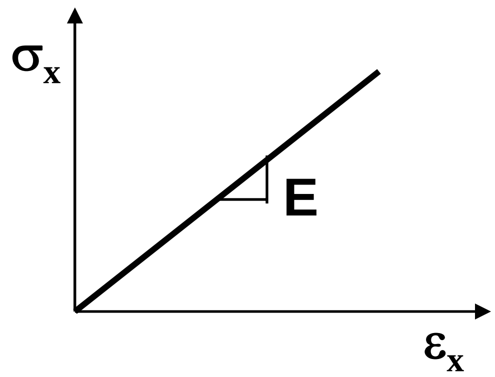
Linear Elasticity in Isotropic Materials

> Young's modulus, E

- The ratio of axial stress to axial strain, under uniaxial loading
- Typical units in solids: GPa = 10^9 Pa
- Typical values – 100 GPa in solids, less in polymers





$$\sigma_x = E\varepsilon_x$$ (for uniaxial loading)

$$\varepsilon_x = \Delta L / L$$

Linear Elasticity in Isotropic Materials

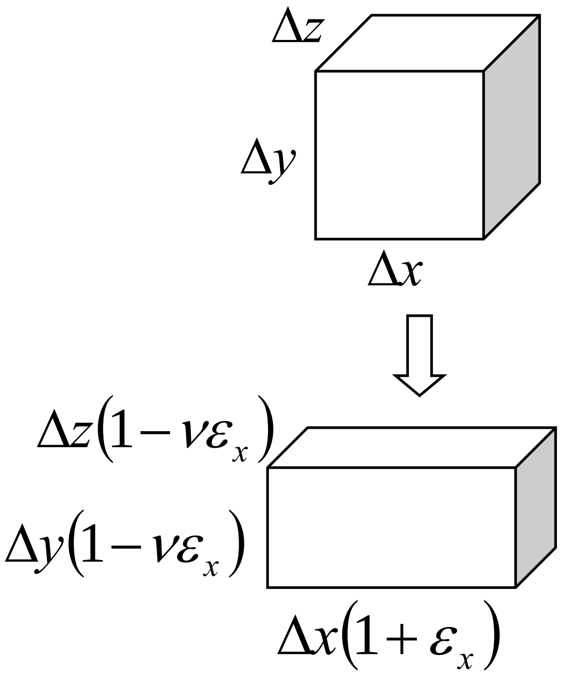
> **Poisson ratio, ν**

- **Some things get narrower in the transverse direction when you extend them axially.**
- **Some things get wider in the transverse direction when you compress them axially.**
- **This is described by the Poisson ratio: the negative ratio of transverse strain to axial strain**
- **Poisson ratio is in the range 0.1 – 0.5 (material dependent)**

$$\varepsilon_y = -\nu \varepsilon_x$$

Poisson's ratio relates to volume change

- Volume change is proportional to (1-2ν)
- As Poisson ratio approaches ½, volume change goes to zero
  - We call such materials incompressible
- Example of incompressible material:
  - Rubber



$$\Delta V = \Delta x \Delta y \Delta z (1+\varepsilon_x)(1-\nu\varepsilon_x)^2 - \Delta x \Delta y \Delta z$$

$$\Downarrow$$

$$\Delta V = \Delta x \Delta y \Delta z (1-2\nu)\varepsilon_x$$

Isotropic Linear Elasticity

- For a general case of loading, the constitutive relationships between stress and elastic strain are as follows
- 6 equations, one for each normal stress and shear stress

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

Shear modulus G is given by $G = \dfrac{E}{2(1+\nu)}$

Other Elastic Constants

> **Other elastic constants in isotropic materials can always be expressed in terms of the Young's modulus and Poisson ratio**

- **Shear modulus G**
- **Bulk modulus (inverse of compressibility)**

$$K = \frac{E}{3(1-2\nu)}$$

Plane stress

- Special case: when all stresses are confined to a single plane Often seen in thin films on substrates (will discuss origin of these stresses shortly)
- Zero normal stress in z direction ($\sigma_z = 0$)
- No constraint on normal strain in z, ε_z

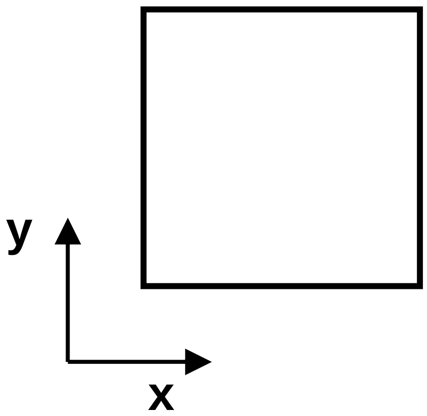
$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

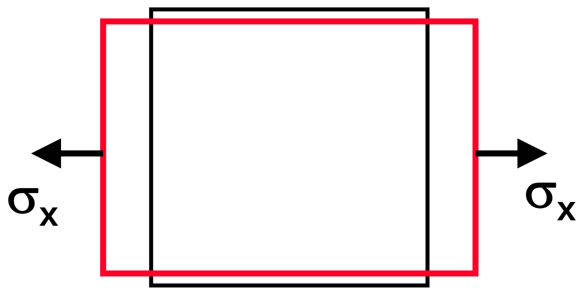
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right) = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right)$$

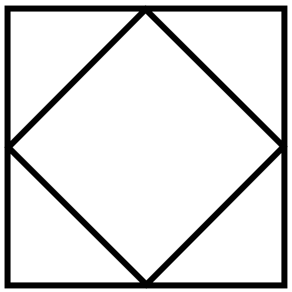
often get insight about these from boundary conditions

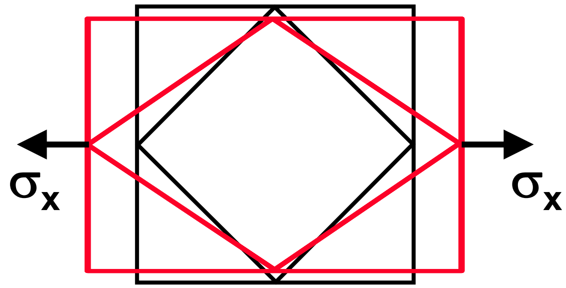
Plane stress: directional dependence





Here, principal axes are in x and y.

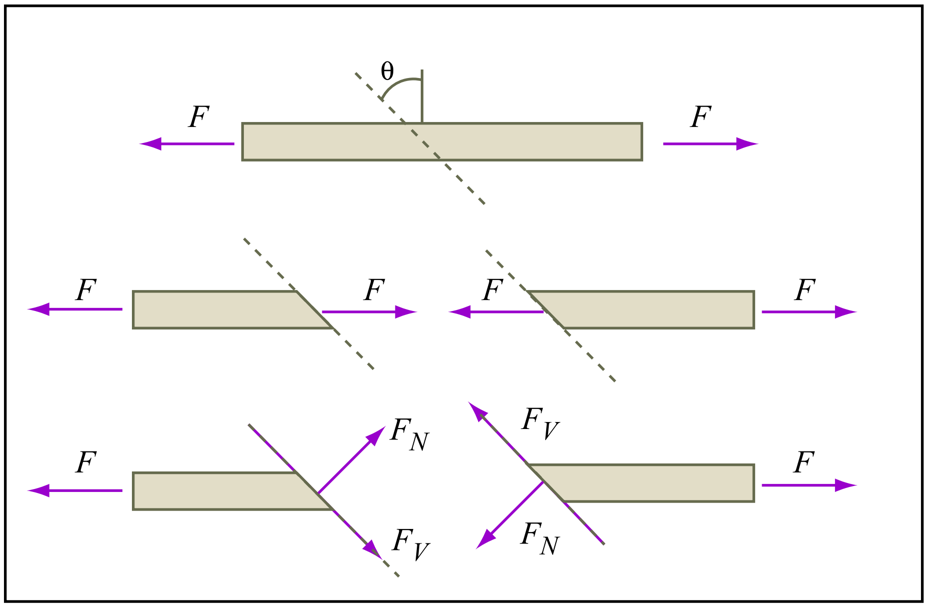




- Principal axes: those directions in which the load appears to be entirely normal stresses (no shear)
- In general, there are shear stresses in other directions

Stresses on Inclined Sections

> **Can resolve axial forces into normal and shear forces on a tilted plane**



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$$F_N = F\cos\theta$$

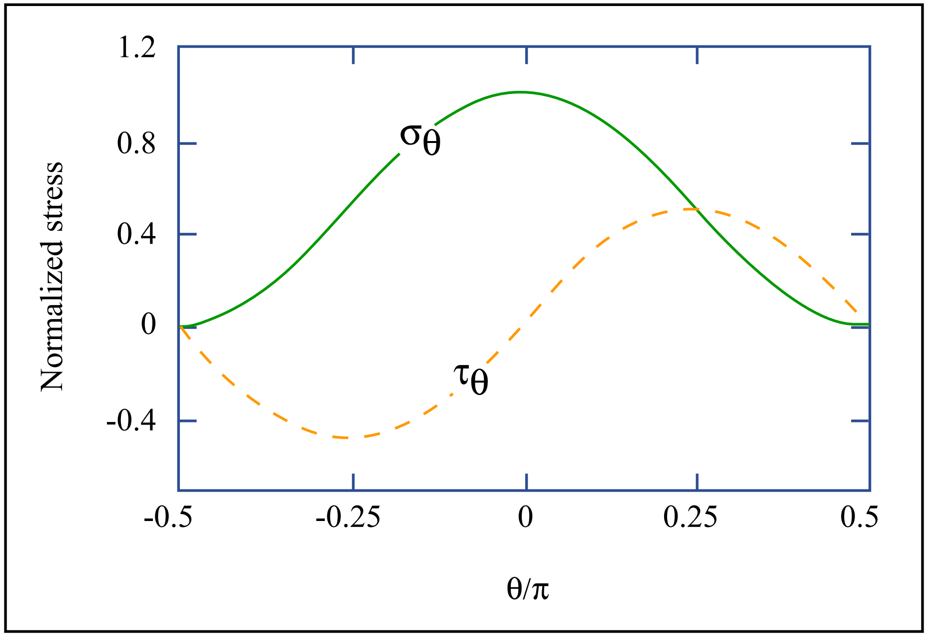
$$F_V = F\sin\theta$$

$$\text{Area} = \frac{A}{\cos\theta}$$

$$\sigma_\theta = \frac{F}{A}\cos^2\theta$$

$$\tau_\theta = \frac{F}{A}\cos\theta\sin\theta$$

Resultant stresses vary with angle



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Failure in shear occurs at an angle of 45 degrees

Special case: biaxial stress

> **A special case of plane stress**

- **Stresses σ_x and σ_y along principal axes are equal**
- **Strains ε_x and ε_y along principal axes are equal**

> **Leads to definition of biaxial modulus**

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

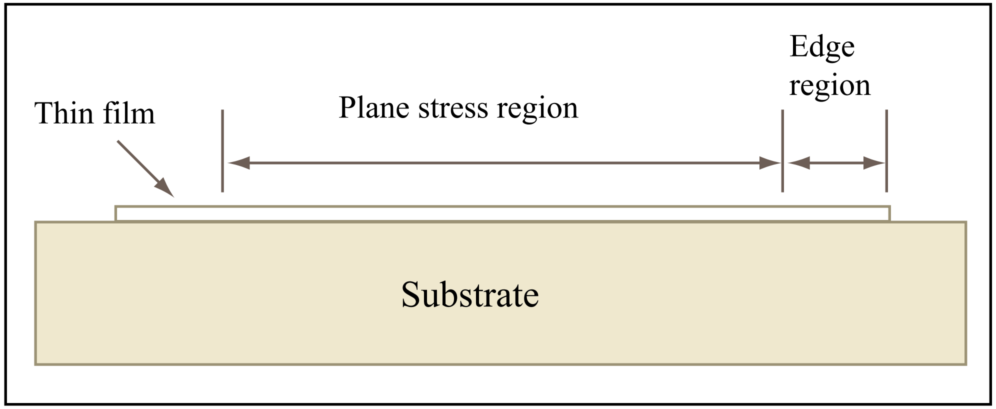
$$\varepsilon = \frac{1}{E}(1-\nu)\sigma$$

$$\sigma = \frac{E}{(1-\nu)}\varepsilon$$

$$\text{Biaxial modulus} = \frac{E}{(1-\nu)}$$

Thin Film Stress

- A thin film on a substrate can have residual stress
  - Intrinsic stress
  - Thermal stress
- Mostly well-described as a plane stress



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Types of strain

> What we have just talked about is elastic strain

- Strains caused by loading; returns to undeformed configuration when load is removed
- Described by the isotropic equations of linear elasticity

> There are other kinds of strain as well

- Thermal strain, which is related to thermal expansion
- Plastic strain: if you stretch something too far, it doesn't return to its undeformed configuration when the load is removed (permanent component)
- Total strain: the sum of all strains

Thermal expansion

- **Thermal expansion: if you change an object's temperature, its length changes**
- **This is a thermally-induced strain**
- **An unopposed thermal expansion produces a strain, but not a stress**
- **If you oppose the thermal expansion, there will be a stress**
- **Coefficient of thermal expansion,** α_T

$$\varepsilon_x^{thermal}(\Delta T) = \alpha_T \Delta T$$

$$\Downarrow$$

$$\varepsilon_x(T) = \varepsilon_x(T_0) + \alpha_T (T - T_0)$$

and

$$\frac{\Delta \mathrm{V}}{\mathrm{V}} = 3\alpha_T (T - T_0)$$

Thermally Induced Residual Stress

- **If a thin film is adhered to a substrate, mismatch of thermal expansion coefficient between film and substrate can lead to stresses in the film (and, to a lesser degree, stresses in the substrate)**
- **The stresses also set up bending moments**
  - **You care about this if you don't want your wafer to curl up like a saucer or potato chip**
- **And the vertical expansion of the film is also modified**

Thermally Induced Residual Stress

Substrate:

$$\varepsilon_s = -\alpha_{T,s}\Delta T$$

where

$$\Delta T = T_d - T_r$$

Film:

$$\varepsilon_{f,free} = -\alpha_{T,f}\Delta T$$

$$\varepsilon_{f,attached} = -\alpha_{T,s}\Delta T$$

Assuming that the film is much thinner than the substrate, the film's actual strain is whatever the substrate imposes.

Some of the final strain is accounted for by the strain that the film would have if it were free. The remainder, or mismatch strain, will be associated with a stress through constitutive relationships.

Mismatch:

$$\varepsilon_{f,mismatch} = (\alpha_{T,f} - \alpha_{T,s})\Delta T$$

Biaxial stress:

$$\sigma_{f,mismatch} = \frac{E}{(1-\nu)}\varepsilon_{f,mismatch}$$

Intrinsic residual stress

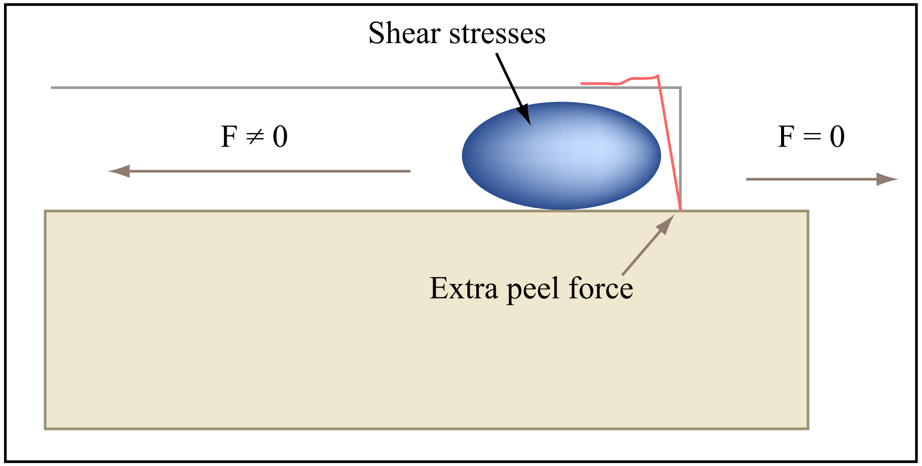
> **Any thin film residual stress that cannot be explained by thermal expansion mismatch is called an intrinsic stress**

> **Sources of intrinsic stress**

- **Deposition far from equilibrium**
- **Secondary grain growth can modify stresses**
- **Ion implantation can produce compressive stress**
- **Substitutional impurities can modify stress**
- **etc….**

Edge effects

- If a bonded thin film is in a state of plane stress due to residual stress created when the film is formed, there are extra stresses at the edges of these films



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- **Linear elasticity in anisotropic materials**
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Storing elastic energy

> **Remember calculating potential energy in physics**

$$U = -\int_{x_i}^{x_f} F_x dx \quad (\text{for example}, U = mgh)$$

> **Deforming a material stores elastic energy**

> **Stress = F/A, strain = ΔL/L**

$$\int_0^{\varepsilon(x,y,z)} \sigma(\varepsilon) d\varepsilon = \ ???$$

> **Together, they contribute 1/length^3: strain energy density at a point in space**

Elastic Energy

- Elastic stored energy **density** is the integral of stress with respect to strain

$$\text{Elastic energy density:} \qquad \tilde{W}(x,y,z) = \int_0^{\varepsilon(x,y,z)} \sigma(\varepsilon)d\varepsilon$$

$$\text{When } \sigma(\varepsilon) = E\varepsilon: \qquad \tilde{W}(x,y,z) = \frac{1}{2}E\left[\varepsilon(x,y,z)\right]^2$$

- The total elastic stored energy is the volume integral of the elastic energy density

$$\text{Total stored elastic energy:} \qquad W = \iiint_{Volume} \tilde{W}(x,y,z)dxdydz$$

- You must know the distribution of stress and strain through a structure in order to find the elastic energy stored in it (next time).

Including Shear Strains

- More generally, the energy density in a linear elastic medium is related to the product of stress and strain (both normal and shear)

$$\text{For axial strains}: \quad \tilde{W} = \frac{1}{2}\sigma\varepsilon$$

$$\text{For shear strains}: \quad \tilde{W} = \frac{1}{2}\tau\gamma$$

This leads to a total elastic strain energy:

$$W = \frac{1}{2}\iiint_{Volume}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)dxdydz$$

Linear elasticity in anisotropic materials

- General case:
  - Stress is a second rank tensor
  - Strain is a second rank tensor
  - Elastic constants form a fourth rank tensor
- There is lots of symmetry in all the tensors
- Can represent stress as a 1 x 6 array and strain as a 1 x 6 array
- The elastic constants form a 6 x 6 array, also with symmetry

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}$$

Stiffness and Compliance

- The matrix of **s**tiffness coefficients, analogous to Young's modulus, are denoted by **C**ij
- The matrix of **c**ompliance coefficients, which is the inverse of Cij, is denoted by **S**ij
- Yes, the notation is cruel
- Some texts use different symbols, but these are quite widely used in the literature

$$\sigma_I = \sum_J C_{IJ} \varepsilon_J$$

and

$$\varepsilon_I = \sum_J S_{IJ} \sigma_J$$

Cubic materials

> Only three independent elastic constants

- C11 = C22 = C33
- C12 = C23 = C31 = C21 = C32 = C13
- C44 = C55 = C66
- All others zero

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}$$

> Values for silicon

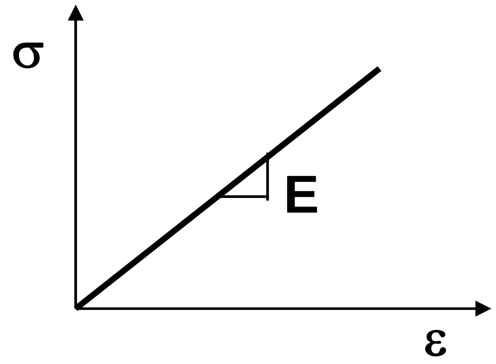
- C11 = 166 GPa
- C12 = 64 GPa
- C44 = 80 GPa

Materials with Lower Symmetry

- **Examples:**
  - **Zinc oxide – 5 elastic constants**
  - **Quartz – 6 elastic constants**
- **These materials come up in piezoelectricity**
- **Otherwise, we can enjoy the fact that most materials we deal with are either isotropic or cubic**

What lies beyond linear elasticity?

> So far, we have assumed linear elasticity.

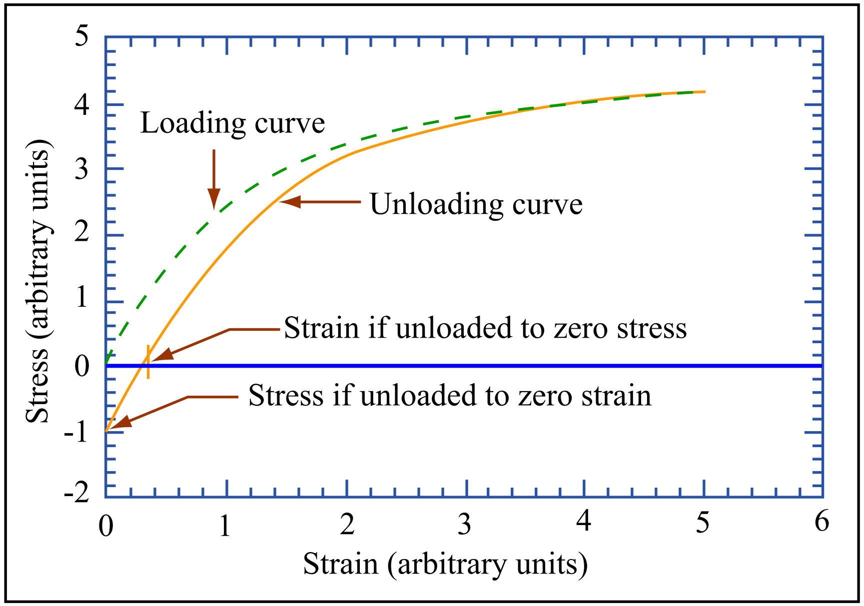


> Linear elasticity fails at large strains

- Some of the deformation becomes permanent (plastic strain)
- Things get stiffer
- Things break

Plastic deformation

- **Beyond the yield point, a plastic material develops a permanent set**
- **This is exploited in the bending and stamping of metals**



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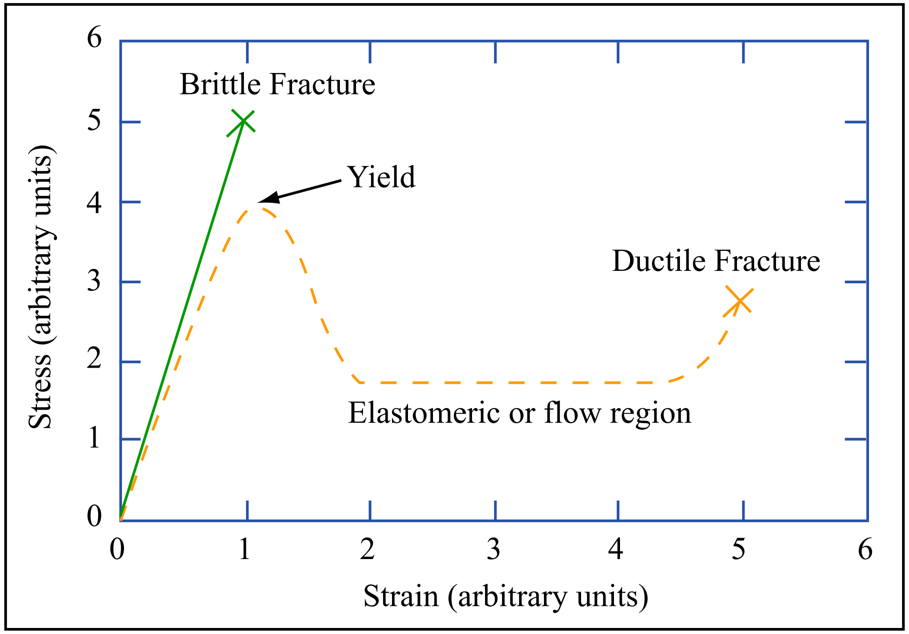
Adapted from Figure 8.8 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 198. ISBN: 9780792372462.

Material behavior at large strain

> **Brittle and ductile materials are very different**



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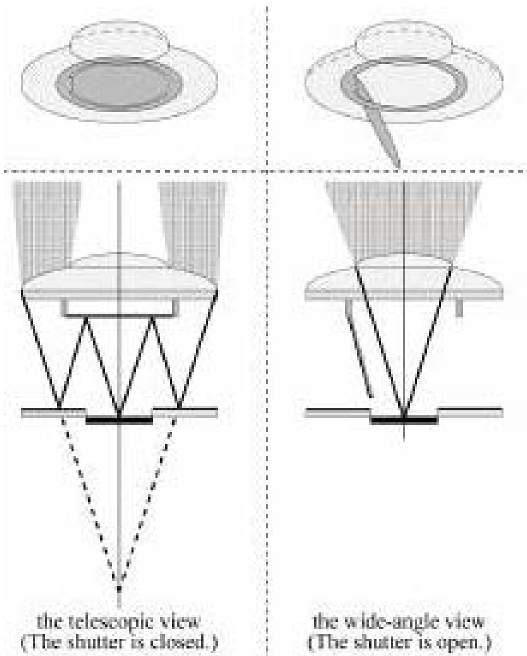
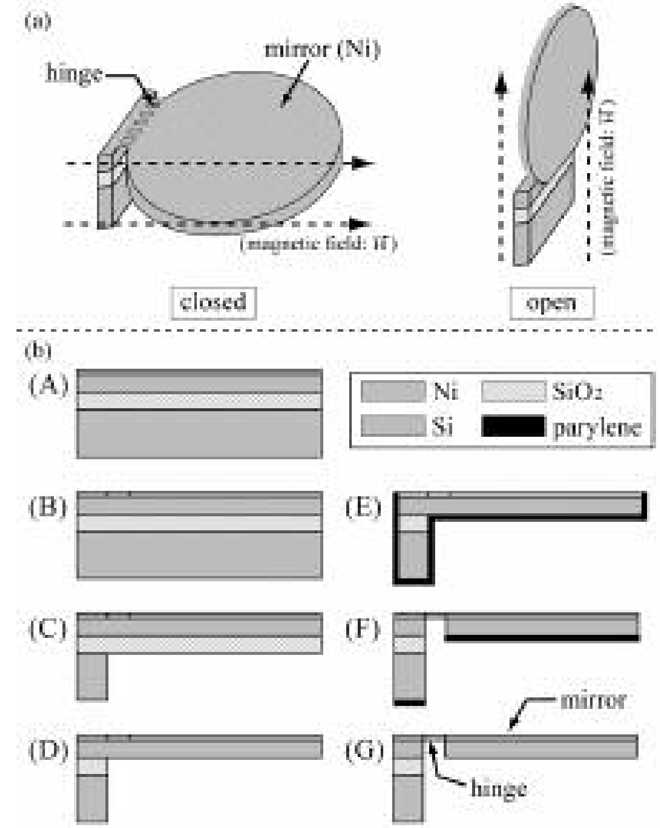
Adapted from Figure 8.7 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 197. ISBN: 9780792372462.

Any thoughts on this device?





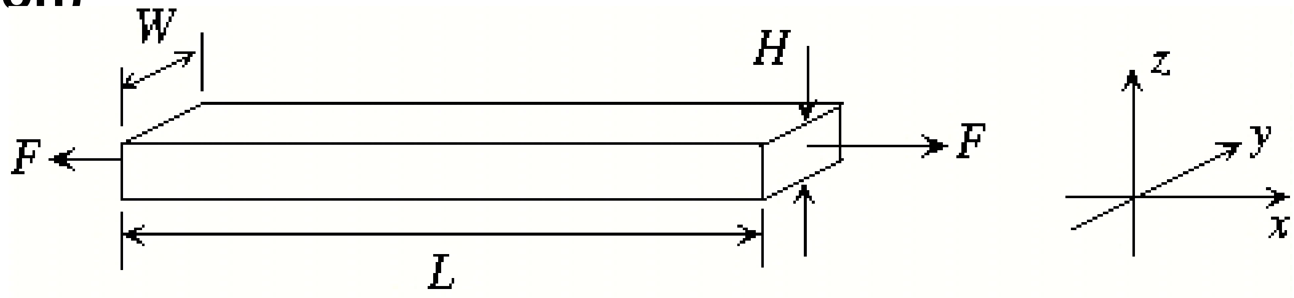
Figures 2, 3, and 4 on pp. 236-237 in: Kinoshita, H., K. Hoshino, K., K. Matsumoto, and I. Shimoyama. "Thin Compound eye Camera with a Zooming Function by Reflective Optics." In *MEMS 2005 Miami: 18th IEEE International Conference on Micro Electro Mechanical Systems: technical digest, Miami Beach, Florida, USA, Jan. 30-Feb. 3, 2005*. Piscataway, NJ: IEEE, 2005, pp. 235-238. ISBN: 9780780387324. © 2005 IEEE.

Outline

- > **Overview**
- > **Some definitions**
  - **Stress**
  - **Strain**
- > **Isotropic materials**
  - **Constitutive equations of linear elasticity**
  - **Plane stress**
  - **Thin films: residual and thermal stress**
- > **A few important things**
  - **Linear elasticity in anisotropic materials**
  - **Behavior at large strains**
- > **Using this to find the stiffness of structures**

A simple example: axially loaded beams

- In equilibrium, force is uniform; hence stress is inversely proportional to area (as long as area changes slowly with position)



Geometry:

$$\sigma = \frac{F}{A} = \frac{F}{WH} \text{ and } \varepsilon = \frac{\Delta \mathrm{L}}{\mathrm{L}}$$

Uniaxial stress:

$$\sigma = \varepsilon E$$

$$\frac{F}{WH} = E\frac{\Delta L}{L}$$

$$F = \frac{EWH}{L}\Delta L$$

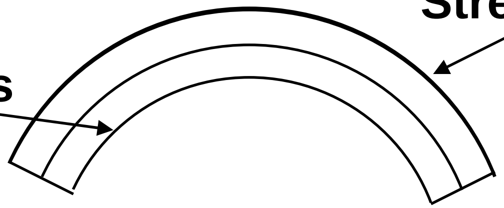
$$F = k\Delta L \Rightarrow k = \frac{EWH}{L}$$

Plug in for L=100 μm, W=5 μm, H=1 μm, E=160 GPa:

k=8000 N/m

Another example: bending of beams and plates

- Stress and strain underlie bending, too
- Unlike uniaxial tension, where stress and strain are uniform, bending of beams and plates is all about how the spatially varying stress and strain contribute to an overall deformation.



- Next time!