Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.685 Electric Machines

Problem Set 6 Solutions — October 17, 2013

Problem 1: Induction Motor

1. The winding plan is shown in Figure 1. Each phase half has two columns (one for each side of the coil). The right hand column is the sum of all turns in that particular slot. See that each phase has two sets of coils. To make it fit on the page, I have wrapped the whole thing into two columns.
2. For the next three parts I wrote a little Matlab script which is attached. Note that the winding factor is obtained by weight averaging the pitch factors of all of the coils by their number of turns. The breadth factor of this winding is one, since all coils have the same axis.

Slot	A	A'	B	B'	C	C'	Tot
1	6					12	18
2	12					6	18
3	18						18
4	18						18
5	12			6			18
6	6			12			18
7				18			18
8				18			18
9				12	6		18
10				6	12		18
11					18		18
12					18		18
13		6			12		18
14		12			6		18
15		18					18
16		18					18
17		12	6				18
18		6	12				18
19			18				18
20			18				18
21			12			6	18
22			6			12	18
23						18	18
24						18	18

Slot	A	A'	B	B'	C	C'	Tot
25	6					12	18
26	12					6	18
27	18						18
28	18						18
29	12			6			18
30	6			12			18
31				18			18
32				18			18
33				12	6		18
34				6	12		18
35					18		18
36					18		18
37		6			12		18
38		12			6		18
39		18					18
40		18					18
41		12	6				18
42		6	12				18
43			18				18
44			18				18
45			12			6	18
46			6			12	18
47						18	18
48						18	18

Figure 1: Winding Pattern

```
Problem Set 6, Problem 1
Total Number of Turns =  144
Winding Factors
kw1 = 0.935908
kw5 = 0.103875  kw7 = 0.0253335
kwm = 0.935908  kwp = -0.935908
Magnetizing Reactance = 96.4565 Ohms
Fundamental Flux Density = 0.699206 T (Peak)
```

Problem 2: This is the permanent magnet retarder problem. We might assume that the shuttle current is:

$$K_R = \mathrm{Re}\left\{\underline{K}_r e^{-jkx}\right\}$$

In the shuttle frame of reference this is:

$$K_R = \mathrm{Re}\left\{\underline{K}_r e^{-jk(x'-ut)}\right\}$$

Ampere's Law applied to this case is:

$$g\frac{\partial H_y}{\partial x} = \underline{K}_R$$

This means that transverse magnetic field is:

$$\underline{H}_y = \frac{j}{kg}\underline{K}_R + H_m$$

where $\underline{H}_m$ is field from the permanent magnets.

Electric field affecting the shuttle is:

$$E'_z = v \times B_y = u\mu_0\left(\frac{j}{kg}\underline{K}_R + \underline{H}_m\right)$$

So that current in the shuttle is:

$$\underline{K}_R = \sigma_s\mu_0 u\left(\frac{j}{kg}\underline{K}_R + \underline{H}_m\right)$$

Or, current in the shuttle is:

$$\underline{K}_R = \frac{\sigma_s u B_m}{\left(1 - j\frac{\sigma_s u \mu_0}{kg}\right)}$$

If $\underline{K}_R$ is in the z direction and B is in the y direction, $K \times B$ is in the -x direction, so force per unit area is:

$$T_x = -\frac{1}{2}\frac{\sigma_s u B_m^2}{1 + \left(\frac{\sigma_s u \mu_0}{kg}\right)^2}$$

Problem 3: In these solutions, I am using a value for gap g that is actually the distance from the magnets to the upper boundary. This is different (by a constant) from what is shown in the problem set: $g(\text{solution}) = g(\text{problem set} - h$. To do a translation from what is here, feel free to just add h to g wherever it appears.

If the problem can be assumed to be one-dimensional, fields are all in the +y direction and:

$$H_g g + H_m h = 0$$

Since flux density is uniform:

$$B = \mu_0 H_g = \mu_m H_m + B_r$$

It is straightforward to find that:

$$\mu_0 H_g \left(1 + \mu_r \frac{g}{h}\right) = B_r$$

where $\mu_r = \frac{\mu_m}{\mu_0}$, then flux density in the system is:

$$B_y = \frac{B_r}{1 + \mu_r \frac{g}{h}} = B_r \frac{h}{h + \mu_r g}$$

Now relax the geometry to have arbitrary gap and magnet dimensions. This is a two-region problem. Call 'Region 1' to be above the magnet: $0 < y < g$, and 'Region 2 is the layer that has the magnets: $-h < y < 0$. We pick the top of the magnets to be $y = 0$ for convenience. Magnetic fields in the two regions can be expressed as: In region 1:

$$\begin{aligned} H_y &= \left(Ae^{-ky} + Be^{ky}\right) \cos kx \\ H_x &= \left(Ae^{-ky} - Be^{ky}\right) \sin kx \end{aligned}$$

In region 2:

$$\begin{aligned} H_y &= \left(Ce^{-ky} + De^{ky}\right) \cos kx \\ H_x &= \left(Ce^{-ky} - De^{ky}\right) \sin kx \end{aligned}$$

This coordinate system was chosen to make the algebra easier. At the top and bottom ferromagnetic boundaries the x-directed field must vanish:

$$\begin{aligned} Ae^{-kg} - Be^{kg} &= 0 \\ Ce^{kh} - De^{-kh} &= 0 \end{aligned}$$

which means:

$$\begin{aligned} B &= Ae^{-2kg} \\ D &= Ce^{2kh} \end{aligned}$$

The other two boundary conditions are that magnetic field parallel to the boundary must be continuous and so must be flux density across the boundary between regions 1 and 2:

$$\begin{aligned} A\left(1-e^{-2kg}\right) &= C\left(1-e^{2kh}\right) \\ \mu_0 A\left(1+e^{-2kg}\right) &= \mu_m C\left(1+e^{2kh}\right)+B_r \end{aligned}$$

The rest is algebra, resulting in expressions for the two constants in the gap region:

$$\begin{aligned} A &= \frac{\frac{1}{\mu_0}B_r\left(1-e^{2kh}\right)}{\left(1-e^{2kh}\right)\left(1+e^{-2kg}\right)-\mu_r\left(1-e^{-2kg}\right)\left(1+e^{2kh}\right)} \\ B &= \frac{\frac{1}{\mu_0}B_r\left(1-e^{2kh}\right)e^{-2kg}}{\left(1-e^{2kh}\right)\left(1+e^{-2kg}\right)-\mu_r\left(1-e^{-2kg}\right)\left(1+e^{2kh}\right)} \end{aligned}$$

Then at $y=g$,

$$H_y = \frac{2}{\mu_0}B_r\frac{\left(1-e^{2kh}\right)e^{-kg}}{\left(1-e^{2kh}\right)\left(1+e^{-2kg}\right)-\mu_r\left(1-e^{-2kg}\right)\left(1+e^{2kh}\right)}\cos kx$$

Just as a check, note that as its argument goes to a small number, $e^x \rightarrow 1+x$, and the magnetic field does indeed approach the small wavenumber limit.

$$H_y = \frac{2}{\mu_0}B_r\frac{-2kh}{-2kh\times 2-\mu_r\times 2kg\times 2} = \frac{B_r}{\mu_0}\frac{h}{h+\mu_r g}$$

Finally, if the gap becomes very large $e^{-kg}\rightarrow 0$, $B\rightarrow 0$ and

$$A = \frac{B_r}{\mu_0}\frac{1-e^{2kh}}{\left(1-e^{2kh}\right)-\mu_r\left(1+e^{2kh}\right)}$$

the magnetic fields are:

$$\begin{aligned} H_y &= \frac{B_r}{\mu_0}\frac{\left(1-e^{2kh}\right)e^{-ky}}{\left(1-e^{2kh}\right)-\mu_r\left(1+e^{2kh}\right)}\cos kx \\ H_x &= \frac{B_r}{\mu_0}\frac{\left(1-e^{2kh}\right)e^{-ky}}{\left(1-e^{2kh}\right)-\mu_r\left(1+e^{2kh}\right)}\sin kx \end{aligned}$$

Problem 4 This problem is one that can be worked using surface impedance techniques. Note that in the region above the moving sheet, complex amplitudes of the magnetic field components will be:

$$\begin{aligned} \underline{H}_x^{(u)} &= H_- e^{-ky} e^{-jkx} \\ \underline{H}_y^{(u)} &= -jH_- e^{-ky} e^{-jkx} \end{aligned}$$

which means that the ratio of fields is:

$$\Gamma_u = \frac{\underline{H}_y}{\underline{H}_x} = -j$$

There will be a current induced in the moving conductor: assuming that the slip frequency is $\omega_s = s\omega$, that current will be:

$$\underline{K}_z = -\mu_0 \sigma_s \frac{\omega_s}{k} \underline{H}_y = -R\underline{H}_y$$

where we use the symbol R to denote the normalized velocity.

Then, since magnetic field at the bottom of the sheet must be:

$$\underline{H}_x^{(\ell)} = \underline{H}_x^{(u)} + \underline{K}_z$$

then the ratio of fields at the underside of the moving sheet is:

$$\Gamma_\ell = \frac{-j}{1 + jR}$$

Now, as we showed in class, it is possible to translate this ratio of fields from just below the moving sheet to just above the driving current sheet. The translation is just like the derivation in class, but note that the displacement is toward *negative* y, so the sign of the sinh terms is reversed:

$$\Gamma_s = j\frac{-j \sinh kg + \Gamma_\ell \cosh kg}{j \cosh kg - j\Gamma_\ell sinhkg}$$

Now the two components of the Maxwell Stress Tensor can be found immediately by:

$$\begin{aligned}
< T_{xy} > &= \frac{1}{2}\mu_0 \mathrm{Re}\left\{\underline{H}_x \underline{H}_y^*\right\} = \frac{\mu_0}{2}|\underline{H}_x|^2 \mathrm{Re}\left\{\Gamma_s\right\} \\
< T_{yy} > &= \frac{1}{2}\mu_0 |\underline{H}_y|^2 - \frac{1}{2}\mu_0 |\underline{H}_x|^2 = -\frac{\mu_0}{2}|\underline{H}_x|^2 \left(1 - |\Gamma_s|^2\right)
\end{aligned}$$

since the normal vector to the moving sheet is in the $-y$ direction, the sign of T_{xy} must be reversed. An example of lift and propulsive force, normalized to $\frac{1}{2}\mu_0 |H_x|^2$ is shown in Figure 2

Appendix: Script for Problem 1

```
% 6.685 Problem Set 6, Problem 1, 2013

% dimensions
Vll = 480;                  % line-line voltage
R = .0254*5.7/2;            % rotor radius
L = .0254*6;                % rotor length
g = .0254*.0185;            % air-gap
p=2;                        % number of pole pairs
om = 2*pi*60;               % frequency
N_s = [6 12 18 18 12 6];      % turns/coil
N_c = [17 15 13 11 9 7];    % coil throw
```

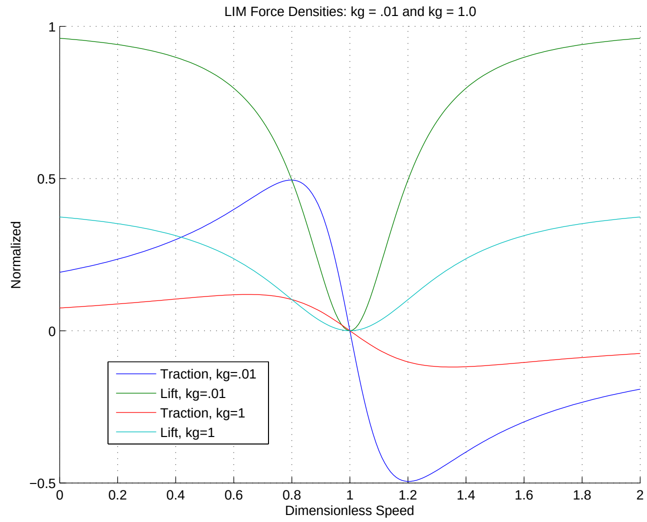

Figure 2: Propulsion and Lift

```
Na = 2*sum(N_s);              % total number of turns
Vph = Vll/sqrt(3);            % phase voltage

muzero= pi*4e-7;
gamma = 2*pi/24;                 % slot angle

kw = sum(N_s .* sin(N_c .*gamma/2)) / sum(N_s);
kw5 = sum(N_s .* sin(N_c .* 5*gamma/2)) / sum(N_s);
kw7 = sum(N_s .* sin(N_c .* 7*gamma/2)) / sum(N_s);
kwm = sum(N_s .* sin(N_c .* 23*gamma/2)) / sum(N_s);
kwp = sum(N_s .* sin(N_c .* 25*gamma/2)) / sum(N_s);

La = (3/2)*(4/pi)*(muzero*R*L/(p^2 *g)) * Na^2 * kw ^2;
Xa = om*La;
B1 = (p/om)*sqrt(2)*Vph/(2*R*L*Na*kw);

fprintf('Problem Set 6, Problem 1\n')
fprintf('Total Number of Turns = %4.0f\n', Na)
fprintf('Winding Factors\n')
fprintf('kw1 = %g \n', kw);
fprintf('kw5 = %g  kw7 = %g\n', kw5, kw7)
fprintf('kwm = %g  kwp = %g\n', kwm, kwp)
fprintf('Magnetizing Reactance = %g Ohms\n', Xa)
fprintf('Fundamental Flux Density = %g T (Peak)\n', B1)
```

```
% open linear motor

kg = [.01 1];            % normalized gap
R0 = 5;                  % Mag Reynold's number at stall
S = 0:.01:2;             % speed in Magnetic arbitrary Number units
R = R0 .* (1-S);         % magnetic 'reynolds number'

figure(1)
clf;
hold on
kg = .01;
gamat = -j ./ (1 + j .* R);
gamaz = j .* (-j * sinh(kg) + gamat .* cosh(kg)) ./( j*cosh(kg) - gamat .* sinh(kg));
Txy1 = real(gamaz);
Tyy1 = 1 - abs(gamaz) .^2;
kg = 1;
gamat = -j ./ (1 + j .* R);
gamaz = j .* (-j * sinh(kg) + gamat .* cosh(kg)) ./( j*cosh(kg) - gamat .* sinh(kg));
Txy2 = real(gamaz);
Tyy2 = 1 - abs(gamaz) .^2;
plot(S, -Txy1, S, Tyy1, S, -Txy2, S, Tyy2)
title('LIM Force Densities: kg = .01 and kg = 1.0')
ylabel('Normalized')
xlabel('Dimensionless Speed')
legend('Traction, kg=.01', 'Lift, kg=.01', 'Traction, kg=1', 'Lift, kg=1')
grid on
```