6.642 Continuum Electromechanics
Fall 2008

Mid-Term - Solutions 2008

Prof. Markus Zahn MIT OpenCourseWare

Problem 1

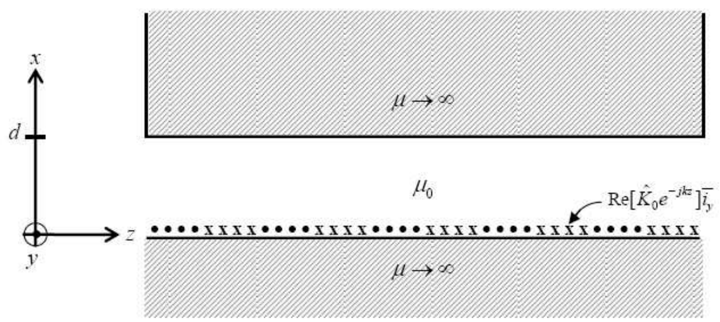

Figure 1: A current sheet at $x = 0$ generates a magnetic field for $0 < x < d$ between two infinite magnetic permeability regions for $x < 0$ and $x > d$.

A current sheet $\text{Re}\left[\hat{K}_0 e^{-jkz}\right]\bar{i}_y$ is placed on the $x = 0$ surface of a material with infinite magnetic permeability ($\mu \to \infty$) for $x < 0$. Another infinite magnetic permeability material extends from $d < x < \infty$. Free space with magnetic permeability μ_0 extends over the region $0 < x < d$.
The magnetic field(H_x)- magnetic scalar potential (χ) relations for the planar layer below

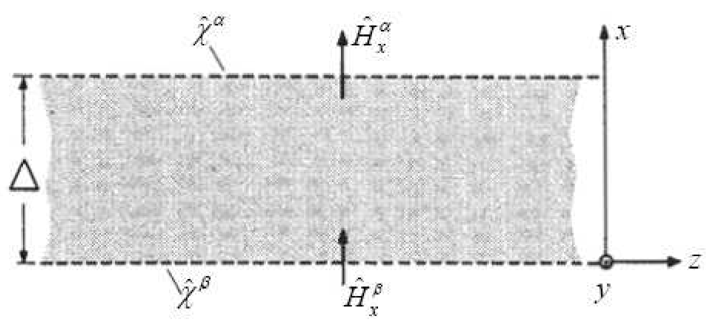

Figure 2: Planar layer used to determine general magnetic field/magnetic scalar potential relationships.

for variables of the form
$\chi(x, z) = \text{Re}\left[\hat{\chi}(x)e^{-jkz}\right]$
are

$$\begin{bmatrix} \hat{H}_x^\alpha \\ \\ \hat{H}_x^\beta \end{bmatrix} = k \begin{bmatrix} \coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ \\ \dfrac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\chi}_\alpha \\ \\ \hat{\chi}_\beta \end{bmatrix}$$

Where $\overline{H}(x,z) = -\nabla\chi(x,z) = \text{Re}\left[\left(\hat{H}_x(x)\bar{i}_x + \hat{H}_z(x)\bar{i}_z\right)e^{-jkz}\right]$
There is no magnetic field dependence on y.

A

<u>Question:</u> What are the boundary conditions on the magnetic field at the $x = 0_+$ and $x = d_-$ surfaces? What are the values of the magnetic scalar potential $\hat{\chi}(x = 0_+)$and$\hat{\chi}(x = d_-)$?

<u>Solution:</u>

$$H_z\left(x = 0_+\right) = -K_y = -Re\left[\hat{K}_0 e^{-jkz}\right] = -\left.\frac{\partial\chi}{\partial z}\right|_{x=0_+}$$

$$jk\hat{\chi}\left(x = 0_+\right) = -\hat{K}_0 \Rightarrow \hat{\chi}\left(x = 0_+\right) = \frac{-\hat{K}_0}{jk} = \frac{j\hat{K}_0}{k}$$

$H_z\left(x = d_-\right) = 0 \Rightarrow \hat{\chi}\left(x = d_-\right) = 0$

B

<u>Question:</u> What are the complex amplitudes of the magnetic field $\overline{H}(x,z)$ at $x = 0_+$ and at $x = d_-$?

<u>Solution:</u>

$$\begin{aligned}
\hat{H}_x\left(x = d_-\right) &= k\left[-\coth kd\cancelto{0}{\hat{\chi}\left(x = d_-\right)} + \frac{1}{\sinh kd}\hat{\chi}\left(x = 0_+\right)\right] \\
&= \frac{k}{\sinh kd}\frac{j\hat{K}_0}{k} = \frac{j\hat{K}_0}{\sinh kd} \\
\hat{H}_z\left(x = d_-\right) &= 0 \\
\hat{H}_z\left(x = 0_+\right) &= k\left[-\frac{1}{\sinh kd}\cancelto{0}{\hat{\chi}\left(x = d_-\right)} + \coth kd\hat{\chi}\left(x = 0_+\right)\right] \\
&= k\coth kd\frac{j\hat{K}_0}{k} = j\hat{K}_0\coth kd \\
\hat{H}_z\left(x = 0_+\right) &= -\hat{K}_0
\end{aligned}$$

C

<u>Question:</u> What is the magnetic force per unit area (on a wave length $2\pi/k$)$\overline{F}$ on the infinite magnetic permeability layer that extends $d < x < \infty$?

<u>Solution:</u>

i) Put Maxwell Stress Tensor surfaces at $x = d_-$ at coordinate y and $y + 2\pi/k$ to extend to $x = +\infty$.

$$\frac{f_x}{area} = -\left.T_{xx}\right|_{x=d_-} = -\frac{\mu_0}{2}\left.\left(H_x^2 - \cancelto{0}{H_y^2} - \cancelto{0}{H_z^2}\right)\right|_{x=d_-} = -\frac{\mu_0}{2}\left.H_x^2\right|_{x=d_-}$$

$$\left\langle\frac{f_x}{area}\right\rangle = -\frac{\mu_0}{4}\left|\hat{H}_x\left(x = d_-\right)\right|^2 = -\frac{\mu_0\left|\hat{K}_0\right|^2}{4\sinh^2 kd}$$

$$\frac{f_z}{area} = -\left.T_{zx}\right|_{x=d_-} = -\mu_0 \cancelto{0}{H_z\left(x = d_-\right)} H_x\left(x = d_-\right) = 0$$

$$\frac{f_y}{area} = -\left.T_{yx}\right|_{x=d_-} = -\mu_0 \cancelto{0}{H_y\left(x = d_-\right)} H_x\left(x = d_-\right) = 0$$

ii) Alternate Surface at $x = 0_+$ extending to $x = +\infty$.

$$\frac{f_x}{area} = -\left.T_{xx}\right|_{x=0_+} = -\frac{\mu_0}{2}\left.\left(H_x^2 - \cancelto{0}{H_y^2} - H_z^2\right)\right|_{x=0_+} = -\frac{\mu_0}{2}\left.\left(H_x^2 - H_z^2\right)\right|_{x=0_+}$$

$$\left\langle\frac{f_x}{area}\right\rangle = -\frac{\mu_0}{4}\left.\left[\left|\hat{H}_x\right|^2 - \left|\hat{H}_z\right|^2\right]\right|_{x=0_+}$$

$$= -\frac{\mu_0}{4}\left[\coth^2 kd - 1\right]\left|\hat{K}_0\right|^2$$

$$= \frac{-\mu_0\left|\hat{K}_0\right|^2}{4\sinh^2 kd}$$

$$\frac{f_z}{area} = -\left.T_{zx}\right|_{x=0_+} = -\mu_0 H_z\left(x = 0_+\right) H_x\left(x = 0_+\right)$$

$$\frac{\langle f_z\rangle}{area} = -\frac{\mu_0}{2} Re\left[\hat{H}_z^*\left(x = 0_+\right)\hat{H}_x\left(x = 0_+\right)\right]$$

$$= -\frac{\mu_0}{2} Re\left[-\hat{K}_0^* j\hat{K}_0 \coth kd\right]$$

$$= \frac{\mu_0}{2}\left|\hat{K}_0\right|^2 \coth kd Re\left[j\right] = 0$$

$$\frac{f_y}{area} = -\left.T_{yx}\right|_{x=0_+} = -\mu_0 \cancelto{0}{H_y\left(x = 0_+\right)} H_x\left(x = 0_+\right) = 0$$

Problem 2

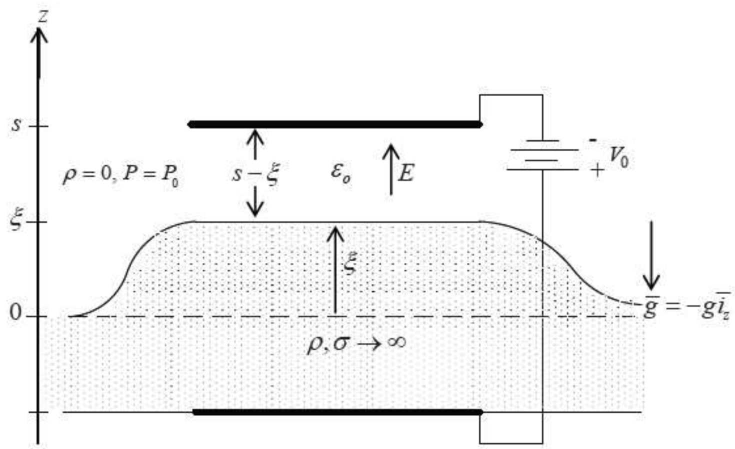

Figure 3: A perfectly conductiong incompressible liquid partially fills the gap between parallel plate electronics stressed by voltage V_0

A perfectly conducting incompressible liquid ($\sigma \to \infty$) with mass density ρ partially fills the gap between parallel plate electrodes stressed by voltage V_0. The applied voltage lifts the fluid interface between the parallel plate electrodes by a height ξ where $\xi < s$. The upper electrode in free space is at $z = s$. When the applied voltage is zero the fluid interface is located at $z = 0$. The region outside the liquid is free space with permittivity ϵ_0, mass density of zero ($\rho = 0$), and atmospheric pressure P_0. The gravitational acceleration is $\overline{g} = -g\overline{i}_z$ and surface tension effects are negligible.

A

Question: What is the electric field for $\xi < z < s$ between the upper electrode at $z = s$ and the perfectly conducting fluid interface at $z = \xi$?
Solution:

$$E_z = \frac{V_0}{s - \xi}$$

B

Question: What is the fluid pressure $p(\xi_-)$ just below the interface at $z = \xi_-$?
Solution:

$$P(\xi_-) - P_0 + T_{zz} = 0$$
$$T_{zz} = \frac{\epsilon_0}{2} E_z^2 = \frac{\epsilon_0}{2}\left(\frac{V_0}{s-\xi}\right)^2$$
$$P(\xi_-) = P_0 - \frac{\epsilon_0}{2}\left(\frac{V_0}{s-\xi}\right)^2$$

C

Question: Find an expression that relates liquid rise $\xi (\xi < s)$ to voltage V_0 and other given parameters.
Solution:

Applying Bernoulli's law at $z = \xi$ and $z = 0$ interfaces within the perfectly conducting fluid where $\overline{f}_{ext} = -\nabla \mathcal{E} = 0 \Rightarrow \mathcal{E} = 0$

$P(\xi_-) + \rho g \xi = P_0 = P_0 - \frac{\epsilon_0}{2}\left(\frac{V_0}{s-\xi}\right)^2 + \rho g \xi$

$\rho g \xi = \frac{\epsilon_0}{2}\left(\frac{V_0}{s-\xi}\right)^2$

$\xi (s-\xi)^2 = \frac{\epsilon_0}{2\rho g} V_0^2$

D

Question: At what voltage is $\xi = \dfrac{s}{2}$?

Solution:

For $\xi = \frac{s}{2}$

$V_0^2 = \frac{2\rho g}{\epsilon_0}\left(\frac{s}{2}\right)^3 = \frac{\rho g s^3}{4\epsilon_0}$

$V_0 = \frac{1}{2}\left[\frac{\rho g s^3}{\epsilon_0}\right]^{1/2}$

Problem 3

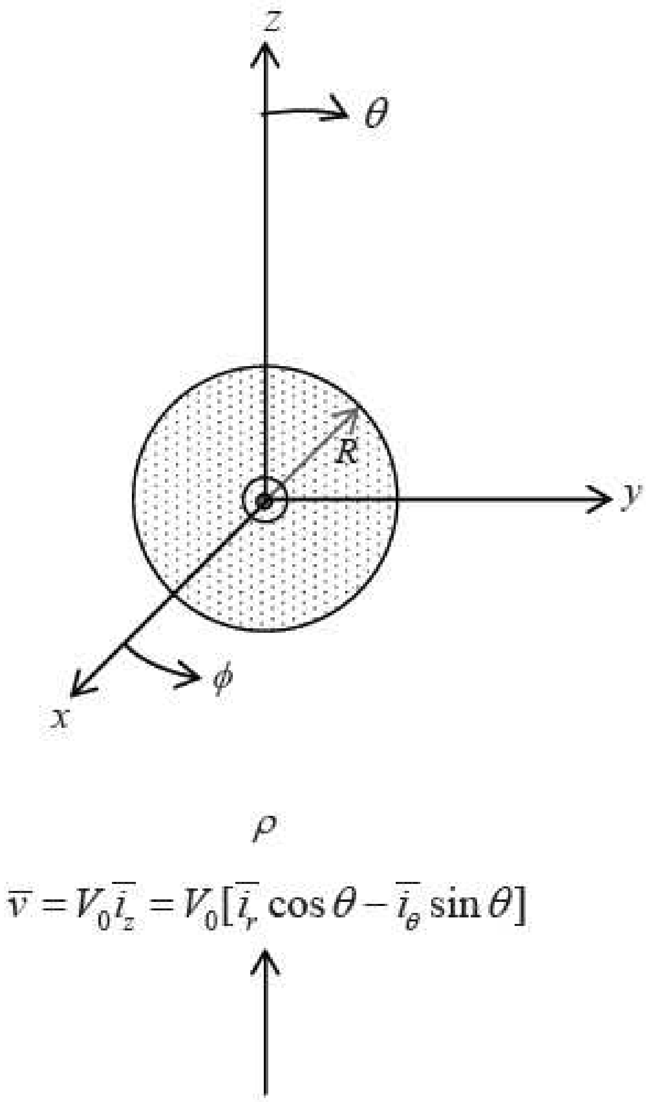

Figure 4: An inviscid incompressible z directed uniform flow from infinity is incident on a sphere of radius R.

An inviscid incompressible liquid with mass density ρ has uniform irrotational flow ($\nabla \times \overline{\nu} = 0$). The flow at $r = \infty$ is uniform and z directed

$$\overline{\nu} = V_0\overline{i_z} = V_0\left[\overline{i_r}\cos\theta - \overline{i_\theta}\sin\theta\right]$$

The flow is incident on a solid sphere of radius R. The inviscid liquid can flow along the sphere so that $\nu_\theta\left(r = R_+\right) \neq 0$ but cannot penetrate the surface so that $\nu_r\left(r = R_+\right) = 0$. Because the irrotational flow has $\nabla \times \overline{\nu} = 0$, a velocity scalar potential Φ can be defined, $\overline{\nu} = -\nabla\Phi$. Because the fluid is also incompressible, $\nabla \bullet \overline{\nu} = 0$, the velocity scalar potential for $r > R$ obeys Laplace's equation, $\nabla^2\Phi = 0$ where $\Phi\left(r, \theta\right)$ does not depend on angle ϕ. The flow does not vary with time and gravity effects are negligible.

A

Question: What are the boundary conditions on the velocity scalar potential at $r = R_+$ and at $r = \infty$?

Solution:

$v_r(r = R_+) = -\left.\frac{\partial \Phi}{\partial r}\right|_{r=R} = 0$

$\overline{v}(r \to \infty) = V_0 \overline{i_z} = V_0 \left(\overline{i_r}\cos\theta - \overline{i_\theta}\sin\theta\right) = -\frac{\partial \Phi}{\partial z}\overline{i_z}$

$\Phi = -V_0 z = -V_0 r\cos\theta$

B

Question: Solve for the velocity scalar potential $\Phi(r,\theta)$.

Solution:

$\Phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta \qquad r > R$

$\Phi(r \Rightarrow \infty, \theta) = -V_0 r\cos\theta = Ar\cos\theta$

$A = -V_0$

$\left.\frac{\partial \Phi}{\partial r}\right|_{r=R} = 0 = \left(A - \frac{2B}{R^3}\right)\cos\theta$

$B = \frac{AR^3}{2} = -\frac{V_0 R^3}{2}$

$\Phi(r,\theta) = -V_0\left(r + \frac{R^3}{2r^2}\right)\cos\theta \qquad r > R$

C

Question: Solve for the velocity field $\overline{v}(r,\theta)$ for $r > R$.

Solution:

$$\begin{aligned}\overline{v} = -\nabla\Phi &= -\left[\frac{\partial \Phi}{\partial r}\overline{i_r} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\overline{i_\theta}\right]\\ &= V_0\left[\left(1 - \frac{R^3}{r^3}\right)\cos\theta\overline{i_r} - \left(1 + \frac{R^3}{2r^3}\right)\sin\theta\overline{i_\theta}\right] \qquad r > R\end{aligned}$$

D

Question: What is the magnitude of the velocity $|\overline{v}(r,\theta)|$?

Solution:

$|\overline{v}| = |V_0|\left[\left(1 - \frac{R^3}{r^3}\right)^2\cos^2\theta\overline{i_r} + \left(1 + \frac{R^3}{2r^3}\right)^2\sin^2\theta\overline{i_\theta}\right]^{1/2}$

E

Question: If the pressure at $r = R_+$ and $\theta = 0$ is P_0, what is the pressure at $r = R_+$,$\theta = \pi/2$?

Solution:

$P + \frac{1}{2}\rho\left|\overline{v}\right|^2 = \text{constant}$

$P\left(R_+, \theta = 0\right) + \frac{1}{2}\rho\left|\overline{v}\left(R, \theta = 0\right)\right|^2 = P_0 + \frac{1}{2}\left(0\right)^2 = P_0 = \text{constant}$

$P\left(R_+, \theta = \frac{\pi}{2}\right) + \frac{1}{2}\rho\left|V_0\right|^2\left(\frac{3}{2}\right)^2 = P_0$

$P\left(R_+, \theta = \frac{\pi}{2}\right) = P_0 - \frac{9}{8}\rho\left|V_0\right|^2$

F

Question: What is the equation for the velocity streamlines?

Solution:

$$\frac{dr}{r d\theta} = \frac{v_r}{v_\theta} = \frac{\cancel{V_0}\left(1 - \frac{R^3}{r^3}\right)\cos\theta}{-\cancel{V_0}\left(1 + \frac{R^3}{2r^3}\right)\sin\theta}$$

$$\frac{\left(1 + \frac{R^3}{r^3}\right)dr}{r\left(1 - \frac{R^3}{r^3}\right)} = \frac{-\cos\theta d\theta}{\sin\theta}$$

$\frac{1}{2}ln\left[r^2\left(1 - \frac{R^3}{r^3}\right)\right] = -ln\left[\sin\theta\right] + \text{costant}$

$ln\left[\sin^2\theta\left(r^2\left(1 - \frac{R^3}{r^3}\right)\right)\right] = \text{constant}$

$\sin^2\left(\frac{r^2}{R^2} - \frac{R}{r}\right) = C$

G

Question: For the velocity streamline that passes through the point $x = 0$, $y = y_0$, $z = 0$ equivalent to $r = y_0$, $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, for what value of y does the streamline pass through when $x = 0$ and $z = -\infty$, equivalent to $r = \infty$, $\theta = \pi$, $\phi = \frac{\pi}{2}$? Find y when $y_0 = R$ and when $y_0 = 2R$.

Solution:

For $\left(r = y_0, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}\right), C = \left(\frac{y_0}{R}\right)^2 - \frac{R}{y_0}$

For $\left(r = \infty, \theta = \pi, \phi = \frac{\pi}{2}\right), \frac{r^2\sin^2\theta}{R^2} = C = \left(\frac{y_0}{R}^2\right) - \frac{R}{y_0}$

$y = r\sin\theta\sin\phi|_{\theta=\pi,\phi=\frac{\pi}{2},r=\infty} = r\sin\theta$

$y^2 = y_0^2 - R^3/y_0 \Rightarrow y = \left[y_0^2 - R^3/y_0\right]^{1/2}$

For $y_0 = 2R$, $\quad y^2 = 4R^2 - \frac{R^3}{2R} = 3.5R^2$

$y = \sqrt{3.5}R \approx 1.871R$

For $y_0 = R, \quad y^2 = R^2 - R^3/R = 0 \Rightarrow y = 0 \sum_2^1$