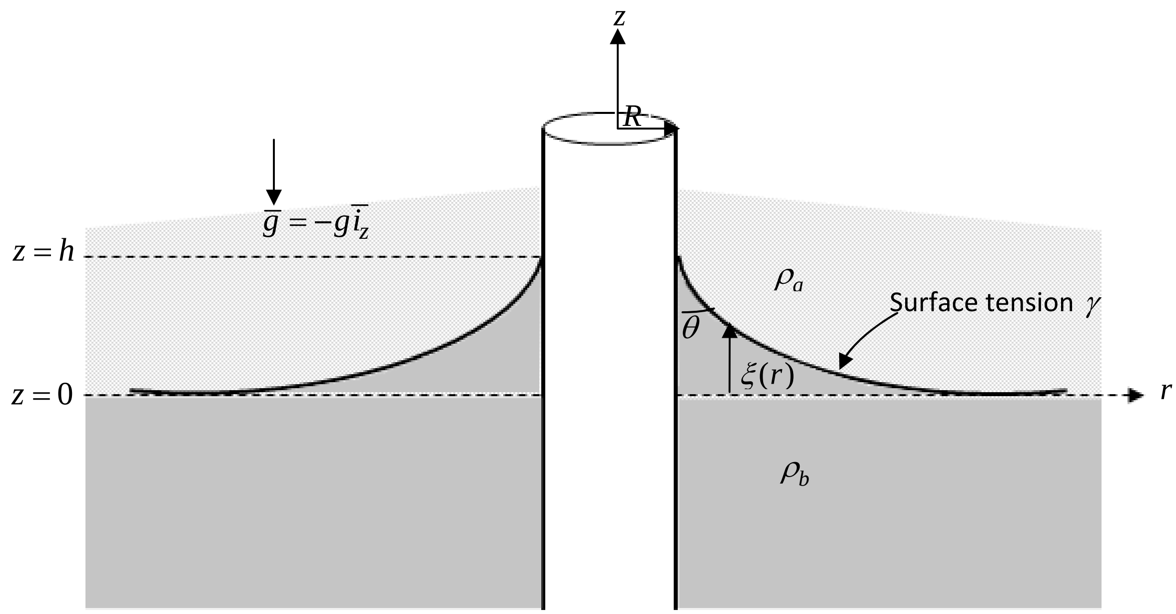
6.642 Continuum Electromechanics
Fall 2008

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.642 Continuum Electromechanics
December 10, 2008

Formula sheets are located after page 6.

1. (33 points)



Two superposed fluids surround and wet a cylindrical rod of radius R. The interfacial surface tension is γ and fluid/rod contact angle is θ. The lower fluid has mass density ρ_b and the upper fluid has mass density ρ_a where $\rho_b > \rho_a$. The vertical displacement of the fluid interface $\xi(r)$ is a function of the radial position r rising to a height h at the rod surface at $r = R$. Thus the fluid /rod interface at $r = R$ has the interface height h and contact angle relationships

$$\xi(r = R) = h \,, \left.\frac{d\xi}{dr}\right|_{r=R} = -\cot(\theta) \,.$$

We assume that there is no variation with the angle ϕ and that the maximum interfacial displacement h is small enough that a linear analysis for $\xi(r)$ can be assumed. Gravity is $\bar{g} = -g\bar{i}_z$.

A) Far from the cylinder $(r >> R)$ the fluid interface is at $z = 0$. For $r = \infty$ what is the difference in pressures just below and just above the interface,
$\Delta P(r = \infty, z = 0) = P_b(r = \infty, z = 0_-) - P_a(r = \infty, z = 0_+)$?

B) Defining the function $F(r,z) = z - \xi(r)$, the interface between the two fluids is located where $F(r,z) = 0$. To linear terms in $\xi(r)$ what is the unit interfacial normal $\overline{n}$?

C) The surface tension force per unit area is given by $\overline{T_s} = -\gamma(\nabla \bullet \overline{n})\overline{n}$. What is $\overline{T_s}$?

D) Using Bernoulli's law and interfacial force balance the governing linear equation for interfacial shape $\xi(r)$ can be written in the form

$$A(r)\frac{d^2\xi(r)}{dr^2} + B(r)\frac{d\xi(r)}{dr} + C(r,\xi(r)) = 0$$

What are $A(r),\ B(r)$ and $C(r,\xi)$?

E) Taking $\xi(r = R) = h$ and $\xi(r = \infty) = 0$, solve for $\xi(r)$.

Hint: One form of Bessel's equation is:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (p^2 + \alpha^2 x^2)y = 0$$

with solution

$$y(x) = C_1 I_p(\alpha x) + C_2 K_p(\alpha x)$$

where I_p is the modified Bessel function of first kind of order p and K_p is the modified Bessel function of second kind of order p.

F) How is h related to the contact angle θ ?

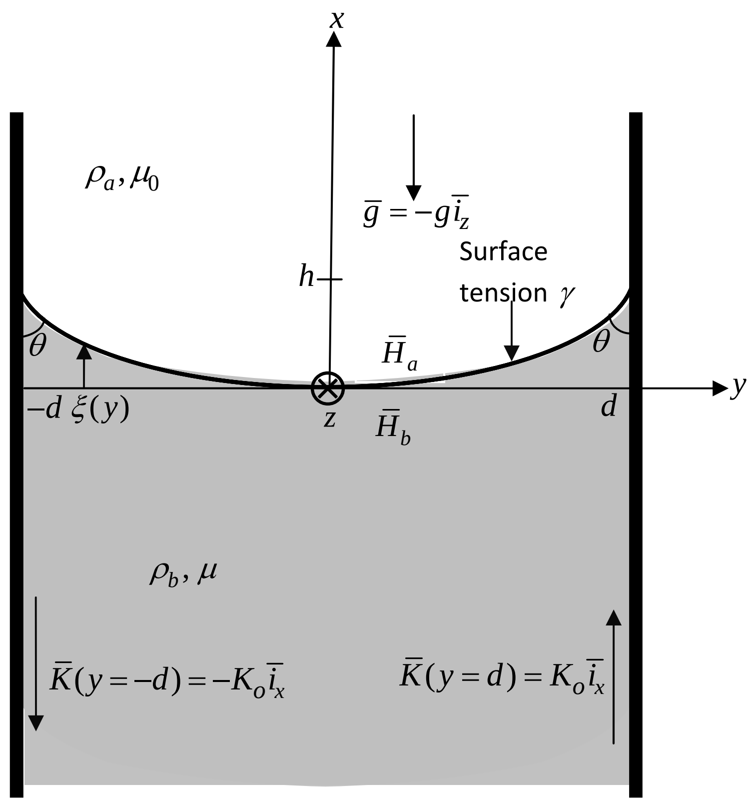
Hints: 1. $$\frac{dI_p(\alpha x)}{dx} = \alpha I_{p+1}(\alpha x) + \frac{pI_p(\alpha x)}{x}$$

2. $$\frac{dK_p(\alpha x)}{dx} = -\alpha K_{p+1}(\alpha x) + \frac{pK_p(\alpha x)}{x}$$

3. $$\frac{dI_0(\alpha x)}{dx} = \alpha I_1(\alpha x)$$

4. $$\frac{dK_0(\alpha x)}{dx} = -\alpha K_1(\alpha x)$$

2. (33 points)



Two superposed and perfectly electrically insulating fluids are contained between vertical plane walls at $y = \pm d$. The fluid interface has surface tension γ and the identical wall contact angles at $y = \pm d$ are θ. The lower fluid is a ferrofluid with mass density ρ_b and magnetic permeability μ and the upper fluid is non-magnetic with mass density ρ_a and magnetic permeability μ_0 with $\rho_b > \rho_a$. The vertical displacement of the fluid interface $\xi(y)$ is a function of position y rising to a height h at $y = \pm d$. Thus the fluid/wall interface at $y = \pm d$ has the interface height h and contact angle relationships

$$\xi(y = d) = \xi(y = -d) = h$$

$$\left.\frac{d\xi}{dx}\right|_{y=d} = \left.\frac{-d\xi}{dy}\right|_{y=-d} = \cot(\theta)$$

The vertical plane walls at $y = \pm d$ are perfectly conducting and carry oppositely directed surface currents $\overline{K}(y = d) = -\overline{K}(y = -d) = K_o\overline{i}_x$

We assume that there is no variation with the z coordinate and that the maximum interfacial displacement h is small enough that a linear analysis for $\xi(y)$ can be assumed. Gravity is $\overline{g} = -g\overline{i}_x$.

A) The magnetic field is assumed to be spatially uniform in both fluids given by

$$\bar{H} = \begin{cases} \bar{H}_a & \text{(upper fluid)} \\ \bar{H}_b & \text{(lower fluid)} \end{cases}$$

What are $\bar{H}_a$ and $\bar{H}_b$ (magnitude and direction)?

B) Defining the function $F(x, y) = x - \xi(y)$, the interface between the two fluids is located where $F(x, y) = 0$. To linear terms in $\xi(y)$ what is the interfacial normal $\bar{n}$?

C) The surface tension force per unit area is given by $\bar{T}_s = -\gamma(\nabla \bullet \bar{n})\bar{n}$. What is $\bar{T}_s$?

D) Using Bernoulli's law within each region find the difference in the pressures just below and above the interface at any position $\xi(y)$,

$\Delta p(y) = P_b(\xi_-(y)) - P_a(\xi_+(y))$

in terms of given parameters and the pressures just below and just above the interface at $y = 0$

$\Delta p(y = 0) = P_b(x = 0_-, y = 0) - P_a(x = 0_+, y = 0)$

Note: It is not yet possible to find the pressure difference $\Delta p(y = 0)$. You will be able to find this in part (f).

E) Using the result of part (D) and interfacial force balance including the magnetic surface force the governing linear equation for $\xi(y)$ can be written in the form

$$\frac{d^2\xi(y)}{dy^2} - A\xi(y) = -B$$

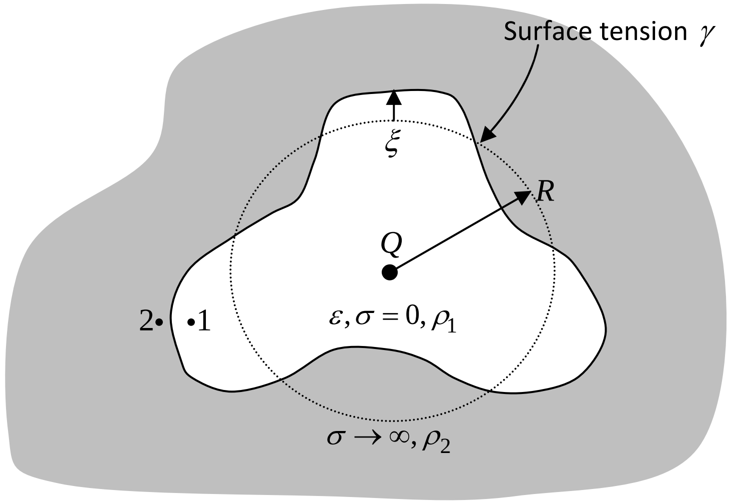
What are A and B ?

F) Taking $\xi(y = d) = \xi(y = -d) = h$ and that $\xi(y = 0) = 0$ solve for $\xi(y)$ in terms of given parameters and $\Delta p(y = 0)$.

G) Solve for the pressure difference just below and just above the interface at $y = 0$, $\Delta p(y = 0)$.

H) How is h related to the contact angle θ ?

3. (34 points)



A point charge Q is located at the center of a perfectly insulating liquid spherical drop with mass density ρ_1 and with dielectric permittivity ε. This drop is surrounded by a perfectly conducting liquid of mass density ρ_2 that extends to $r = \infty$. The point charge Q is fixed to $r = 0$ and cannot move from this position. The fluid interface has surface tension γ. As the interface is radially perturbed by displacement $\xi(\theta,\phi,t) = \mathrm{Re}[\hat{\xi} P_n^m(\cos\theta) e^{j(\omega t - m\phi)}]$ all perturbation variables change as:

Fluid velocity: $\overline{v}(r,\theta,\phi,t) = \mathrm{Re}[(\hat{v}_r(r)\overline{i}_r + \hat{v}_\theta(r)\overline{i}_\theta + \hat{v}_\phi(r)\overline{i}_\phi) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}]$ $\quad 0 < r < \infty$ (both regions)

Pressure: $p(r,\theta,\phi,t) = \mathrm{Re}[\hat{p}(r) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}]$ $\quad 0 < r < \infty$ (both regions)

Electric field: $\overline{e}(r,\theta,\phi,t) = \mathrm{Re}[(\hat{e}_r(r)\overline{i}_r + \hat{e}_\theta(r)\overline{i}_\theta + \hat{e}_\phi(r)\overline{i}_\phi) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}]$ $\quad 0 < r < R + \xi$ (inner droplet)

Electric potential: $\overline{e} = -\nabla\Phi,\ \Phi(r,\theta,\phi,t) = \mathrm{Re}[(\hat{\Phi}(r) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}]$ $\quad 0 < r < R + \xi$ (inner droplet)

A position just inside the interface at $r = (R + \xi)_-$ is labeled 1 and just outside the interface at $r = (R + \xi)_+$ is labeled 2.

A) What is the equilibrium electric potential $\Phi(r)$, and electric field $\overline{E}(r) = -\nabla\Phi$ within the inner droplet for $0 < r < R$.

B) What is the equilibrium jump in pressure across the spherical interface $\Delta p(r = R) = p_1(r = R_-) - p_2(r = R_+)$?

C) What boundary condition must the total electric field satisfy at the $r = R + \xi$ interface? Apply this boundary condition to determine the perturbation electric scalar potential complex amplitude $\hat{\Phi}(r = R_-)$ in terms of interfacial displacement complex amplitude $\hat{\xi}$.

D) What are the perturbation pressure complex amplitudes $\hat{p}_1$ and $\hat{p}_2$ at both sides of the $r = R + \xi$ interface in terms of interfacial displacement complex amplitude $\hat{\xi}$.

Hint: Use transfer relations from Tables 7.9.1 and 2.16.3 from Continuum Electromechanics by J.R. Melcher (attached). Take the perturbation velocity at $\boldsymbol{r} = \infty$ and $\boldsymbol{r} = 0$ to be zero.

E) What is the radial component of the perturbation interfacial stress complex amplitude $\hat{T}_{sr}$ due to surface tension in terms of interfacial displacement complex amplitude $\hat{\xi}$?

Hint: See surface tension Table 7.6.2 from Continuum Electromechanics by J.R. Melcher (attached).

F) What is the perturbation radial electric field complex amplitude $\hat{e}_r(r = R\_)$ in terms of $\hat{\Phi}(r = R\_)$? Using the results of part (C) express $\hat{e}_r(r = R\_)$ in terms of $\hat{\xi}$.

Hint: Use transfer relations from Table 2.16.3 from Continuum Electromechanics by J.R. Melcher (attached).

G) Find the dispersion relation. Is the spherical droplet stabilized or destabilized by the electric field from the point charge Q?

H) If (G) is stabilizing, what is the lowest oscillation frequency? If (G) is destabilizing, what is the lowest value of n that is unstable and what is the growth rate of the instability? What value of Q will only have one unstable mode?

6.642 Formula Sheet #1

Cartesian Coordinates (x, y, z)

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \mathbf{i}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{i}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{i}_z\frac{1}{r}\left[\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates (r, θ, ϕ)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial (r^2 A_r)} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times A = \mathbf{i}_r\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$+\mathbf{i}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right] + \mathbf{i}_\phi\frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Cartesian		*Cylindrical*		*Spherical*
x	=	$r\cos\phi$	=	$r\sin\theta\cos\phi$
y	=	$r\sin\phi$	=	$r\sin\theta\sin\phi$
z	=	z	=	$r\cos\theta$
$\mathbf{i}_x$	=	$\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi$	=	$\sin\theta\cos\phi\mathbf{i}_r + \cos\theta\cos\phi\mathbf{i}_\theta - \sin\phi\mathbf{i}_\phi$
$\mathbf{i}_y$	=	$\sin\phi\mathbf{i}_r + \cos\phi\mathbf{i}_\phi$	=	$\sin\theta\sin\phi\mathbf{i}_r + \cos\theta\sin\phi\mathbf{i}_\theta + \cos\phi\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$

Cylindrical		*Cartesian*		*Spherical*
r	=	$\sqrt{x^2+y^2}$	=	$r\sin\theta$
ϕ	=	$\tan^{-1}\frac{y}{x}$	=	ϕ
z	=	z	=	$r\cos\theta$
$\mathbf{i}_r$	=	$\cos\phi\mathbf{i}_x + \sin\phi\mathbf{i}_y$	=	$\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_\theta$
$\mathbf{i}_\phi$	=	$-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$	=	$\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$

Spherical		*Cartesian*		*Cylindrical*
r	=	$\sqrt{x^2+y^2+z^2}$	=	$\sqrt{r^2+z^2}$
θ	=	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	=	$\cos^{-1}\frac{z}{\sqrt{r^2+z^2}}$
ϕ	=	$\cot^{-1}\frac{x}{y}$	=	ϕ
$\mathbf{i}_r$	=	$\sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$	=	$\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_z$
$\mathbf{i}_\theta$	=	$\cos\theta\cos\phi\mathbf{i}_x + \cos\theta\sin\phi\mathbf{i}_y - \sin\theta\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_z$
$\mathbf{i}_\phi$	=	$-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$	=	$\mathbf{i}_\phi$

Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems.

6.642 Formula Sheet #1

Vector Identities

$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$\nabla\cdot(\nabla\times\mathbf{A})=0$

$\nabla\times(\nabla f)=0$

$\nabla(fg)=f\nabla g+g\nabla f$

$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$

$\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+(\mathbf{A}\cdot\nabla)f$

$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$

$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

$\nabla\times(f\mathbf{A})=\nabla f\times\mathbf{A}+f\nabla\times\mathbf{A}$

$(\nabla\times\mathbf{A})\times\mathbf{A}=(\mathbf{A}\cdot\nabla)\mathbf{A}-\frac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A})$

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

Integral Theorems

Line Integral of a Gradient

$$\int_a^b \nabla f\cdot d\mathbf{l}=f(b)-f(a)$$

Divergence Theorem:

$$\int_V \nabla\cdot\mathbf{A}dV=\oint_S \mathbf{A}\cdot d\mathbf{S}$$

Corollaries

$$\int_V \nabla f dV=\oint_S f d\mathbf{S}$$

$$\int_V \nabla\times\mathbf{A}dV=-\oint_S \mathbf{A}\times d\mathbf{S}$$

Stokes' Theorem:

$$\oint_L \mathbf{A}\cdot d\mathbf{l}=\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

Corollary

$$\oint_L f d\mathbf{l}=-\int_S \nabla f\times d\mathbf{S}$$

Maxwell's Equations

Integral	*Differential*	*Boundary Conditions*
Faraday's Law		
$\oint_L \mathbf{E}'\cdot d\mathbf{l}=-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$	$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$	$\mathbf{n}\times(\mathbf{E}_2'-\mathbf{E}_1')=0$
Ampere's Law with Maxwell's Displacement Current Correction		
$\oint_L \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}_f\cdot d\mathbf{S}+\frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}$	$\nabla\times\mathbf{H}=\mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial d}$	$\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}_f$
Gauss's Law		
$\oint_S \mathbf{D}\cdot d\mathbf{S}=\int_V \rho_f dV$	$\nabla\cdot\mathbf{D}=\rho_f$	$\mathbf{n}\cdot(\mathbf{D}_2-\mathbf{D}_1)=\sigma_f$
$\oint_S \mathbf{B}\cdot d\mathbf{S}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{n}\cdot(\mathbf{B}_2-\mathbf{B}_1)=0$
Conservation of Charge		
$\oint_S \mathbf{J}_f\cdot d\mathbf{S}+\frac{d}{dt}\int_V \rho_f dV=0$	$\nabla\cdot\mathbf{J}_f+\frac{\partial\rho_f}{\partial t}=0$	$\mathbf{n}\cdot(\mathbf{J}_2-\mathbf{J}_1)+\frac{\partial\sigma_f}{\partial t}=0$

Usual Linear Constitutive Laws

$\mathbf{D}=\epsilon\mathbf{E}$

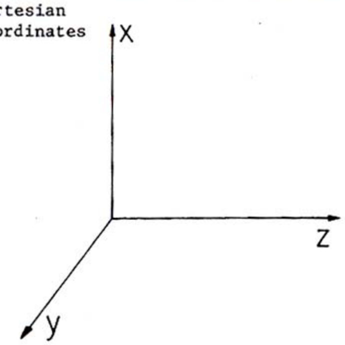
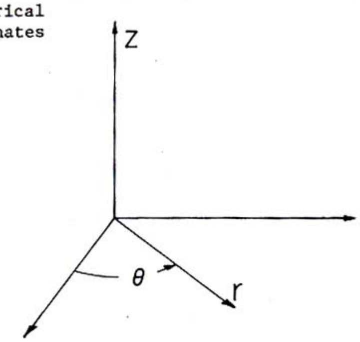
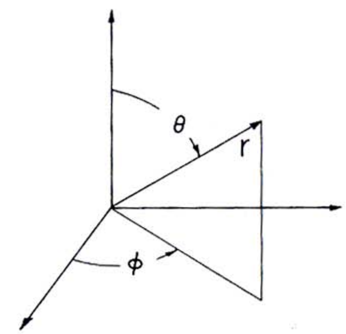
$\mathbf{B}=\mu\mathbf{H}$

$\mathbf{J_f}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})=\sigma\mathbf{E}'$ [Ohm's law for moving media with velocity $\mathbf{v}$]

Physical Constants

Constant	Symbol	Value	units
Speed of light in vacuum	c	$2.9979\times10^8\approx3\times10^8$	m/sec
Elementary electron charge	e	1.602×10^{-19}	coul
Electron rest mass	m_e	9.11×10^{-31}	kg
Electron charge to mass ratio	$\frac{e}{m_e}$	1.76×10^{11}	coul/kg
Proton rest mass	m_p	1.67×10^{-27}	kg
Boltzmann constant	k	1.38×10^{-23}	joule/°K
Gravitation constant	G	6.67×10^{-11}	nt-m^2/(kg)^2
Acceleration of gravity	g	9.807	m/(sec)^2
Permittivity of free space	ϵ_0	$8.854\times10^{-12}\approx\frac{10^{-9}}{36\pi}$	farad/m
Permeability of free space	μ_0	$4\pi\times10^{-7}$	henry/m
Planck's constant	h	6.6256×10^{-34}	joule-sec
Impedance of free space	$\eta_0=\sqrt{\frac{\mu_0}{\epsilon_0}}$	$376.73\approx120\pi$	ohms
Avogadro's number	N	6.023×10^{23}	atoms/mole

APPENDIX A. Differential Operators in Cartesian, Cylindrical and Spherical Coordinates

Operator	Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
			
$(\nabla\cdot\vec{A})$	$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
$\nabla\Phi$	$\frac{\partial\Phi}{\partial x}\vec{i}_x+\frac{\partial\Phi}{\partial y}\vec{i}_y+\frac{\partial\Phi}{\partial z}\vec{i}_z$	$\frac{\partial\Phi}{\partial r}\vec{i}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{i}_\theta+\frac{\partial\Phi}{\partial z}\vec{i}_z$	$\frac{\partial\Phi}{\partial r}\vec{i}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{i}_\theta+\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\vec{i}_\phi$
$(\nabla^2\Phi)$	$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Phi}{\partial r})+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$
$(\nabla\times\vec{A})$	$\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\vec{i}_x$ $+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\vec{i}_y$ $+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\vec{i}_z$	$\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta}-\frac{\partial A_\theta}{\partial z}\right)\vec{i}_r$ $+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\vec{i}_z$	$\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\vec{i}_r$ $+\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\vec{i}_\phi$

Appendix A in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981. ISBN: 9780262131650.

$\nabla^2\vec{A}$	$\left[\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right]\vec{i}_x$ $+\left[\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right]\vec{i}_y$ $+\left[\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}\right]\vec{i}_z$	$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r)\right)+\frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial^2 A_r}{\partial z^2}\right]\vec{i}_r$ $+\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}+\frac{\partial^2 A_\theta}{\partial z^2}\right]\vec{i}_\theta$ $+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A_z}{\partial\theta^2}+\frac{\partial^2 A_z}{\partial z^2}\right]\vec{i}_z$	$\left[\nabla^2 A_r-\frac{2A_r}{r^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}-\frac{2A_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]\vec{i}_r$ $+\left[\nabla^2 A_\theta+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}-\frac{A_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right]\vec{i}_\theta$ $+\left[\nabla^2 A_\phi-\frac{A_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right]\vec{i}_\phi$
$\vec{C}\cdot\nabla\vec{A}$	$\left(C_x\frac{\partial A_x}{\partial x}+C_y\frac{\partial A_x}{\partial y}+C_z\frac{\partial A_x}{\partial z}\right)\vec{i}_x$ $+\left(C_x\frac{\partial A_y}{\partial x}+C_y\frac{\partial A_y}{\partial y}+C_z\frac{\partial A_y}{\partial z}\right)\vec{i}_y$ $+\left(C_x\frac{\partial A_z}{\partial x}+C_y\frac{\partial A_z}{\partial y}+C_z\frac{\partial A_z}{\partial z}\right)\vec{i}_z$	$\left(C_r\frac{\partial A_r}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_r}{\partial\theta}+C_z\frac{\partial A_r}{\partial z}-\frac{C_\theta A_\theta}{r}\right)\vec{i}_r$ $+\left(C_r\frac{\partial A_\theta}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+C_z\frac{\partial A_\theta}{\partial z}+\frac{C_\theta A_r}{r}\right)\vec{i}_\theta$ $+\left(C_r\frac{\partial A_z}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_z}{\partial\theta}+C_z\frac{\partial A_z}{\partial z}\right)\vec{i}_z$	$\left(C_r\frac{\partial A_r}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_r}{\partial\theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{C_\theta A_\theta}{r}-\frac{C_\phi A_\phi}{r}\right)\vec{i}_r$ $+\left(C_r\frac{\partial A_\theta}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}+\frac{C_\theta A_r}{r}-\frac{C_\phi A_\phi\cot\theta}{r}\right)\vec{i}_\theta$ $+\left(C_r\frac{\partial A_\phi}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\phi}{\partial\theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{C_\phi A_r}{r}+\frac{C_\phi A_\theta\cot\theta}{r}\right)\vec{i}_\phi$
$(\vec{\vec{T}}:\nabla\vec{A})$	$T_{xx}\left(\frac{\partial A_x}{\partial x}\right)+T_{yy}\left(\frac{\partial A_y}{\partial y}\right)+T_{zz}\left(\frac{\partial A_z}{\partial z}\right)$ $+T_{xy}\left(\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}\right)$ $+T_{yz}\left(\frac{\partial A_y}{\partial z}+\frac{\partial A_z}{\partial y}\right)+T_{zx}\left(\frac{\partial A_z}{\partial x}+\frac{\partial A_x}{\partial z}\right)$	$T_{rr}\left(\frac{\partial A_r}{\partial r}\right)+T_{\theta\theta}\left(\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{A_r}{r}\right)+T_{zz}\left(\frac{\partial A_z}{\partial z}\right)$ $+T_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{A_\theta}{r}\right)+\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)+T_{\theta z}\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta}+\frac{\partial A_\theta}{\partial z}\right)$ $+T_{rz}\left(\frac{\partial A_z}{\partial r}+\frac{\partial A_r}{\partial z}\right)$	$T_{rr}\left(\frac{\partial A_r}{\partial r}\right)+T_{\theta\theta}\left(\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{A_r}{r}\right)+T_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{A_r}{r}+\frac{A_\theta\cot\theta}{r}\right)$ $+T_{r\theta}\left(\frac{\partial A_\theta}{\partial r}+\frac{1}{r}\frac{\partial A_r}{\partial\theta}-\frac{A_\theta}{r}\right)+T_{r\phi}\left(\frac{\partial A_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{A_\phi}{r}\right)$ $+T_{\theta\phi}\left(\frac{1}{r}\frac{\partial A_\phi}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}-\frac{\cot\theta}{r}A_\phi\right)$
$\nabla\cdot\vec{\vec{T}}$	$\left(\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}\right)\vec{i}_x$ $+\left(\frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}\right)\vec{i}_y$ $+\left(\frac{\partial T_{zx}}{\partial x}+\frac{\partial T_{zy}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\right)\vec{i}_z$	$\left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr})+\frac{1}{r}\frac{\partial}{\partial\theta}T_{r\theta}-\frac{1}{r}T_{\theta\theta}+\frac{\partial T_{rz}}{\partial z}\right)\vec{i}_r$ $+\left(\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{\partial T_{r\theta}}{\partial r}+\frac{2}{r}T_{r\theta}+\frac{\partial T_{\theta z}}{\partial z}\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{zr})+\frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta}+\frac{\partial T_{zz}}{\partial z}\right)\vec{i}_z$	$\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{r\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}\right)\vec{i}_r$ $+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}+\frac{T_{r\theta}}{r}-\frac{\cot\theta}{r}T_{\phi\phi}\right)\vec{i}_\theta$ $+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{T_{r\phi}}{r}+\frac{2\cot\theta}{r}T_{\theta\phi}\right)\vec{i}_\phi$

$T_{\theta\phi}$?

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.642 FORMULA SHEET

1. DIFFERENTIAL OPERATORS IN CYLINDRICAL AND SPHERICAL COORDINATES

If r, ϕ, and z are circular **cylindrical coordinates** and $\hat{\imath}_r$, $\hat{\imath}_\phi$, and $\hat{\imath}_z$ are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\phi \frac{1}{r}\frac{\partial U}{\partial \phi} + \hat{\imath}_z \frac{\partial U}{\partial z}$$

$$\nabla \cdot \vec{A} = div \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = curl \vec{A} = \hat{\imath}_r \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\imath}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\imath}_z \left(\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

If r, θ, and ϕ are **spherical coordinates** and $\hat{\imath}_r$, $\hat{\imath}_\theta$, and $\hat{\imath}_\phi$ are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\theta \frac{1}{r}\frac{\partial U}{\partial \theta} + \hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\nabla \cdot \vec{A} = div \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \vec{A} = curl \vec{A} = \hat{\imath}_r \left(\frac{1}{r\sin\theta}\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right) + \hat{\imath}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\right) + \hat{\imath}_\phi \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

2. SOLUTIONS OF LAPLACE'S EQUATIONS

A. Rectangular coordinates, two dimensions (independent of z):

$$\Phi = e^{kx}(A_1 \sin ky + A_2 \cos ky) + e^{-kx}(B_1 \sin ky + B_2 \cos ky)$$

(or replace e^{kx} and e^{-kx} by $\sinh kx$ and $\cosh kx$).

$$\Phi = Axy + Bx + Cy + D; (k = 0)$$

B. Cylindrical coordinates, two dimensions (independent of z):

$$\Phi = r^n(A_1 \sin n\phi + A_2 \cos n\phi) + r^{-n}(B_1 \sin n\phi + B_2 \cos n\phi)$$

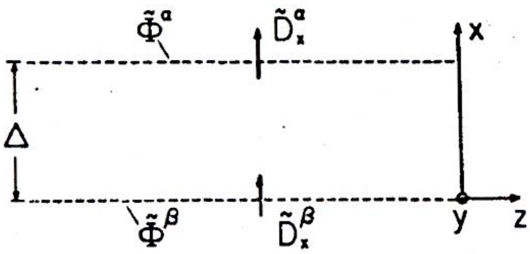
$$\Phi = \ln\frac{R}{r}(A_1\phi + A_2) + B_1\phi + B_2; (n = 0)$$

C. Spherical coordinates, two dimensions (independent of ϕ):

$$\Phi = Ar\cos\theta + \frac{B}{r^2}\cos\theta + \frac{C}{r} + D$$

Table 2.16.1. Flux-potential transfer relations for planar layer in terms of electric potential and normal displacement (Φ, D_x). To obtain magnetic relations, substitute $(\Phi, D_x, \varepsilon) \rightarrow (\Psi, B_x, \mu)$.

Planar layer



$$\begin{bmatrix} \tilde{D}_x^{\alpha} \\ \tilde{D}_x^{\beta} \end{bmatrix} = \varepsilon\gamma \begin{bmatrix} -\coth(\gamma\Delta) & \dfrac{1}{\sinh(\gamma\Delta)} \\ \dfrac{-1}{\sinh(\gamma\Delta)} & \coth(\gamma\Delta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^{\alpha} \\ \tilde{\Phi}^{\beta} \end{bmatrix} \quad \text{(a)}$$

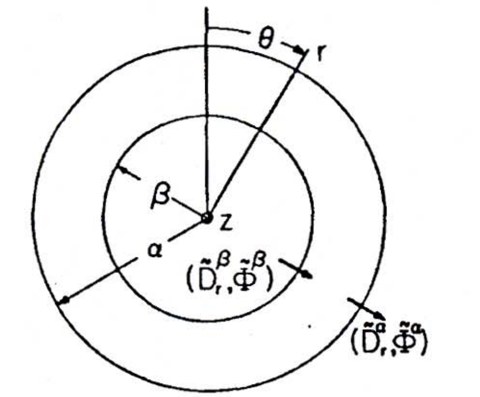
$$\begin{bmatrix} \tilde{\Phi}^{\alpha} \\ \tilde{\Phi}^{\beta} \end{bmatrix} = \frac{1}{\varepsilon\gamma} \begin{bmatrix} -\coth(\gamma\Delta) & \dfrac{1}{\sinh(\gamma\Delta)} \\ \dfrac{-1}{\sinh(\gamma\Delta)} & \coth(\gamma\Delta) \end{bmatrix} \begin{bmatrix} \tilde{D}_x^{\alpha} \\ \tilde{D}_x^{\beta} \end{bmatrix} \quad \text{(b)}$$

$$\Phi = \mathrm{Re}\,\tilde{\Phi}(x,t)\, e^{-j(k_y y + k_z z)}$$

$$\gamma \equiv \sqrt{k_y^2 + k_z^2}$$

Table 2.16.1 in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981, p. 2.33. ISBN: 9780262131650.

Table 2.16.2. Flux-potential relations for cylindrical annulus in terms of electric potential and normal displacement (Φ, D_r). To obtain magnetic relations, substitute $(\Phi, D_r, \varepsilon) \rightarrow (\Psi, B_r, \mu)$.



$$\Phi = \text{Re}\,\tilde{\Phi}(r,t)e^{-j(m\theta + kz)}$$

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \varepsilon \begin{bmatrix} f_m(\beta,\alpha) & g_m(\alpha,\beta) \\ g_m(\beta,\alpha) & f_m(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad \text{(a)}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} F_m(\beta,\alpha) & G_m(\alpha,\beta) \\ G_m(\beta,\alpha) & F_m(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad \text{(b)}$$

$k = 0,\ m = 0$

$$f_o(x,y) = \frac{1}{y}/\ln\left(\frac{x}{y}\right); \quad g_o(x,y) = \frac{1}{x}/\ln\left(\frac{x}{y}\right)$$

$k = 0,\ m = 0$

No inverse

$k = 0,\ m = 1,2,\cdots$

$$f_m(x,y) = \frac{m}{y}\frac{\left[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m\right]}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$g_m(x,y) = \frac{2m}{x}\frac{1}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$k = 0,\ m = 1,2,\cdots$

$$F_m(x,y) = \frac{y}{m}\frac{\left[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m\right]}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$G_m(x,y) = \frac{2y}{m}\frac{1}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$k \neq 0,\ m = 0,1,2\cdots$*

$$f_m(x,y) = \frac{jk[H_m(jkx)J_m'(jky) - J_m(jkx)H_m'(jky)]}{[J_m(jkx)H_m(jky) - J_m(jky)H_m(jkx)]}$$

$$g_m(x,y) = \frac{-2j}{\pi x[J_m(jkx)H_m(jky) - J_m(jky)H_m(jkx)]}$$

$$f_m(x,y) = \frac{k[K_m(kx)I_m'(ky) - I_m(kx)K_m'(ky)]}{[I_m(kx)K_m(ky) - I_m(ky)K_m(kx)]}$$

$$g_m(x,y) = \frac{1}{x[I_m(kx)K_m(ky) - I_m(ky)K_m(kx)]}$$

$k \neq 0,\ m = 0,1,2,\cdots$*

$$F_m(x,y) = \frac{1}{jk}\frac{[J_m'(jkx)H_m(jky) - H_m'(jkx)J_m(jky)]}{[J_m'(jky)H_m'(jkx) - J_m'(jkx)H_m'(jky)]}$$

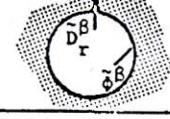
$$G_m(x,y) = \frac{-2}{j\pi k(kx)[J_m'(jky)H_m'(jkx) - J_m'(jkx)H_m'(jky)]}$$

$$F_m(x,y) = \frac{1}{k}\frac{[I_m'(kx)K_m(ky) - K_m'(kx)I_m(ky)]}{[I_m'(ky)K_m'(kx) - I_m'(kx)K_m'(ky)]}$$

$$G_m(x,y) = \frac{1}{k(kx)[I_m'(ky)K_m'(kx) - I_m'(kx)K_m'(ky)]}$$

$\beta \rightarrow 0$ 

$$\tilde{D}_r^\alpha = \varepsilon f_m(0,\alpha)\tilde{\Phi}^\alpha; \quad f_m(0,\alpha) = -\frac{kI_m'(k\alpha)}{I_m(k\alpha)} \quad \text{(c)}$$

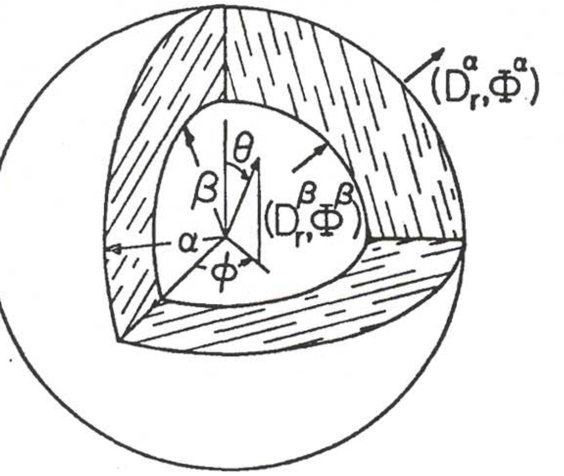
$\alpha \rightarrow \infty$ 

$$\tilde{D}_r^\beta = \varepsilon f_m(\infty,\beta)\tilde{\Phi}^\beta; \quad f_m(\infty,\beta) = -\frac{kK_m'(k\beta)}{K_m(k\beta)} \quad \text{(d)}$$

*See Prob. 2.17.2 for proof that $H_m(jkx)J_m'(jkx) - J_m(jkx)H'_m(jkx) = -2/(\pi kx)$ and $K_m(kx)I_m'(kx) - I_m(kx)K_m'(kx) = 1/kx$ incorporated into g_m and G_m.

Table 2.16.2 in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981, p. 2.35. ISBN: 9780262131650.

Table 2.16.3. Flux-potential transfer relations for spherical shell in terms of electric potential and normal displacement (Φ, D_r). To obtain magnetic relations, substitute $(\Phi, D_r, \varepsilon) \rightarrow (\Psi, B_r, \mu)$.



$$\Phi = \text{Re}\, \tilde{\Phi}(r,t) P_n^m(\cos\theta) e^{-jm\phi}$$

$$P_n^m = (1 - x^2)^{m/2} \frac{d^m P_n}{dx^m}$$

$$P_o = 1, \quad P_1 = x, \quad P_2 = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3 = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4 = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

m	P_o^m	P_1^m	$P_1^m \cos m\phi$	P_2^m	$P_2^m \cos m\phi$	P_3^m	$P_3^m \cos m\phi$
0	1	$\cos\theta$	+ −	$\frac{1}{2}(3\cos^2\theta - 1)$	+ − +	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$	+ − + −
1	0	$\sin\theta$	+ − +	$3\sin\theta\cos\theta$	+ − + − + −	$\frac{3}{2}\sin\theta(5\cos^2\theta - 1)$	+ − + − + − + − +
2	0	0		$3\sin^2\theta$	+ − + − +	$15\sin^2\theta\cos\theta$	+ − + − + − + − + −
3	0	0		0		$15\sin^3\theta$	+ − + − + − +

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \varepsilon \begin{bmatrix} f_n(\beta,\alpha) & g_n(\alpha,\beta) \\ g_n(\beta,\alpha) & f_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad \text{(a)}$$

$$f_n(x,y) = \frac{\left[n\left(\frac{y}{x}\right)^n + (n+1)\left(\frac{x}{y}\right)^{n+1}\right]}{\left[x\left(\frac{x}{y}\right)^n - y\left(\frac{y}{x}\right)^n\right]}$$

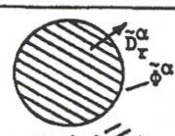
$$g_n(x,y) = \frac{(2n+1)}{x^2\left[\frac{1}{y}\left(\frac{x}{y}\right)^n - \frac{1}{x}\left(\frac{y}{x}\right)^n\right]}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} F_n(\beta,\alpha) & G_n(\alpha,\beta) \\ G_n(\beta,\alpha) & F_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad \text{(b)}$$

$$F_n(x,y) = \frac{y}{x} \frac{\left[\frac{1}{n}\left(\frac{y}{x}\right)^n + \frac{1}{n+1}\left(\frac{x}{y}\right)^{n+1}\right]}{\left[\frac{1}{y}\left(\frac{x}{y}\right)^n - \frac{1}{x}\left(\frac{y}{x}\right)^n\right]}$$

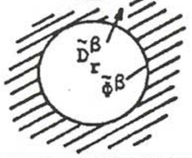
$$G_n(x,y) = \frac{y}{x} \frac{(2n+1)}{n(n+1)} \frac{1}{\left[\frac{1}{y}\left(\frac{x}{y}\right)^n - \frac{1}{x}\left(\frac{y}{x}\right)^n\right]}$$

$\beta \rightarrow 0$



$$\tilde{D}_r^\alpha = -\frac{\varepsilon n}{\alpha} \tilde{\Phi}^\alpha \quad \text{(c)}$$

$\alpha \rightarrow \infty$



$$\tilde{D}_r^\beta = \frac{\varepsilon(n+1)}{\beta} \tilde{\Phi}^\beta \quad \text{(d)}$$

Table 7.6.2. Summary of normal vector and surface tension surface force density for small perturbations from planar, circular cylindrical and spherical equilibria.

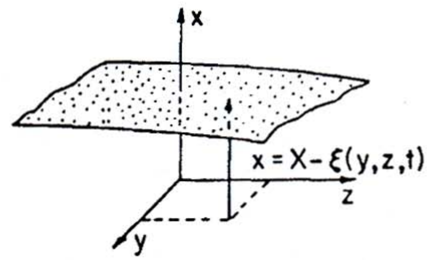
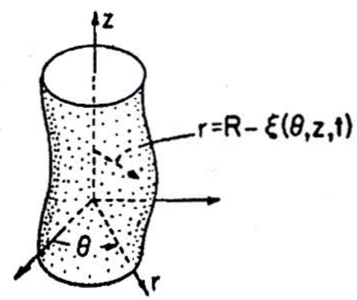
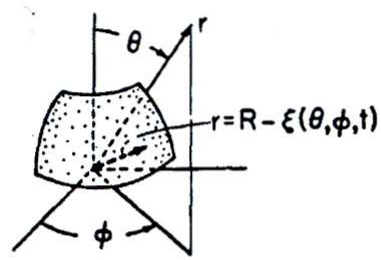
	$\vec{n} = \vec{i}_x - \frac{\partial\xi}{\partial y}\vec{i}_y - \frac{\partial\xi}{\partial z}\vec{i}_z$	(a)
	$(\vec{T}_s)_x = \gamma\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right)$	(b)
	$\xi = \mathrm{Re}\,\tilde{\xi}\exp -j(k_y y + k_z z)$	(c)
	$\tilde{T}_s = -\gamma(k_y^2 + k_z^2)\tilde{\xi}$	(d)
	$\vec{n} = \vec{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\vec{i}_\theta - \frac{\partial\xi}{\partial z}\vec{i}_z$	(e)
	$(\vec{T}_s)_r = \gamma\left[-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right]$	(f)
	$\xi = \mathrm{Re}\,\tilde{\xi}\exp -j(m\theta + kz)$	(g)
	$\tilde{T}_s = \frac{\gamma}{R^2}\left[(1 - m^2) - (kR)^2\right]\tilde{\xi}$	(h)
	$\vec{n} = \vec{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\vec{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\vec{i}_\phi$	(i)
	$(\vec{T}_s)_r = \gamma\left[-\frac{2}{R} + \frac{2\xi}{R^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}\right]$	(j)
	$\xi = \mathrm{Re}\,\tilde{\xi}\,P_n^m(\cos\theta)e^{-jm\phi}$	(k)
	$\tilde{T}_s = -\frac{\gamma}{R^2}(n-1)(n+2)\tilde{\xi}$	(ℓ)

Table 7.6.2 in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981, p. 7.7. ISBN: 9780262131650.

Table 7.9.1. Pressure-velocity relations for perturbations of inviscid fluid.

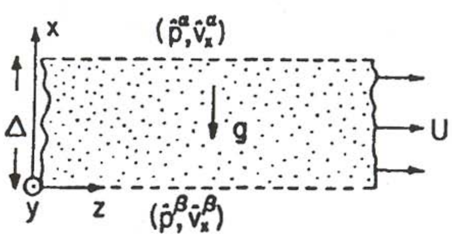
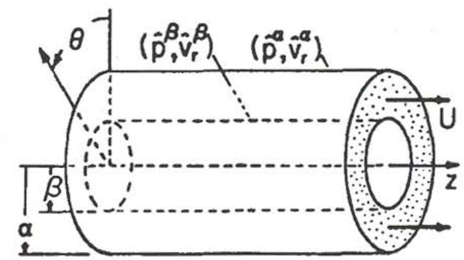
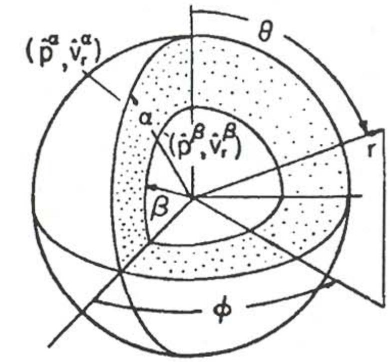
Cartesian	Cylindrical	Spherical
		
$p = \Pi - \frac{1}{2}\rho U^2 - \mathcal{E} - \rho g x + \text{Re}\,\hat{p}(x)e^{j(\omega t - k_y y - k_z z)}$ (a)	$p = \Pi - \frac{1}{2}\rho U^2 - \mathcal{E} + \text{Re}\,\hat{p}(r)e^{j(\omega t - m\theta - kz)}$ (d)	$p = \Pi - \mathcal{E} + \text{Re}\,\hat{p}(r)P_n^m(\cos\theta)e^{j(\omega t - m\phi)}$ (g)
$\hat{p}(x) = j(\omega - k_z U)\rho\hat{\Phi}(r)$ (b)	$\hat{p}(r) = j(\omega - kU)\rho\hat{\Phi}(r)$ (e)	$\hat{p}(r) = j\omega\rho\hat{\Phi}(r)$ (h)
$\begin{bmatrix}\hat{p}^\alpha\\ \hat{p}^\beta\end{bmatrix} = \frac{j(\omega - k_z U)\rho}{\gamma}\begin{bmatrix}-\coth\gamma\Delta & \frac{1}{\sinh\gamma\Delta}\\ \frac{-1}{\sinh\gamma\Delta} & \coth\gamma\Delta\end{bmatrix}\begin{bmatrix}\hat{v}_x^\alpha\\ \hat{v}_x^\beta\end{bmatrix}$ (c) $\gamma \equiv \sqrt{k_y^2 + k_z^2}$	$\begin{bmatrix}\hat{p}^\alpha\\ \hat{p}^\beta\end{bmatrix} = j(\omega - kU)\rho\begin{bmatrix}F_m(\beta,\alpha) & G_m(\alpha,\beta)\\ G_m(\beta,\alpha) & F_m(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\hat{v}_r^\alpha\\ \hat{v}_r^\beta\end{bmatrix}$ (f) (See Table 2.16.2 for F_m and G_m)	$\begin{bmatrix}\hat{p}^\alpha\\ \hat{p}^\beta\end{bmatrix} = j\omega\rho\begin{bmatrix}F_n(\beta,\alpha) & G_n(\alpha,\beta)\\ G_n(\beta,\alpha) & F_n(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\hat{v}_r^\alpha\\ \hat{v}_r^\beta\end{bmatrix}$ (i) (See Table 2.16.3 for F_n and G_n)
Compressible: $\gamma \equiv \sqrt{k_y^2 + k_z^2 - \frac{(\omega - k_z U)^2}{a^2}}$	Compressible; replace $k \to \gamma$ in F_m and G_m: $\gamma \equiv \sqrt{k^2 - \frac{(\omega - kU)^2}{a^2}}$	

Table 7.9.1 in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981, p. 7.12. ISBN: 9780262131650.