

6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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Lecture 10: Solutions to Laplace's Equation In Cartesian Coordinates

I. Poisson's Equation

$$\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla\Phi$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \Rightarrow \nabla \cdot (\nabla\Phi) = \nabla^2\Phi = -\frac{\rho}{\epsilon} \text{ (Poisson's Equation)}$$

II. Particular and Homogeneous Solutions

$$\nabla^2\Phi_p = -\frac{\rho}{\epsilon} \quad \text{Poisson's Equation} \Rightarrow \Phi_p(\vec{r}) = \int_V \frac{\rho(\vec{r}') dV'}{4\pi\epsilon|\vec{r}-\vec{r}'|}$$

$$\nabla^2\Phi_h = 0 \quad \text{Laplace's Equation}$$

$$\nabla^2(\Phi_p + \Phi_h) = -\frac{\rho}{\epsilon}$$

$$\Phi = \Phi_p + \Phi_h \text{ must satisfy boundary conditions}$$

III. Uniqueness of Solutions

Try 2 solutions Φ_a and Φ_b

$$\nabla^2\Phi_a = -\frac{\rho}{\epsilon}$$

$$\nabla^2\Phi_b = -\frac{\rho}{\epsilon}$$

$$\nabla^2(\Phi_a - \Phi_b) = 0$$

$$\Phi_d = \Phi_a - \Phi_b \Rightarrow \nabla^2\Phi_d = 0 \Rightarrow \Phi_d = 0$$

$$\nabla \cdot [\Phi_d \nabla\Phi_d] = \Phi_d \nabla^2\Phi_d + \nabla\Phi_d \cdot \nabla\Phi_d = |\nabla\Phi_d|^2$$

$$\int_V \nabla \cdot [\Phi_d \nabla\Phi_d] dV = \oint_S \Phi_d \nabla\Phi_d \cdot \vec{d}\vec{a} = \int_V |\nabla\Phi_d|^2 dV = 0$$

$$\text{on } S, \Phi_d = 0 \text{ or } \nabla\Phi_d \cdot \vec{d}\vec{a} = 0$$

$$\Phi_d = 0 \Rightarrow \Phi_a = \Phi_b \text{ on } S$$

$$\nabla\Phi_d \cdot \vec{d}\vec{a} = 0 \Rightarrow \frac{\partial\Phi_a}{\partial n} = \frac{\partial\Phi_b}{\partial n} \text{ on } S \Rightarrow E_{na} = E_{nb} \text{ on } S$$

A problem is uniquely posed when the potential or the normal derivative of the potential (normal component of electric field) is specified on the surface surrounding the volume.

IV. Boundary Conditions

1. Gauss' Continuity Condition

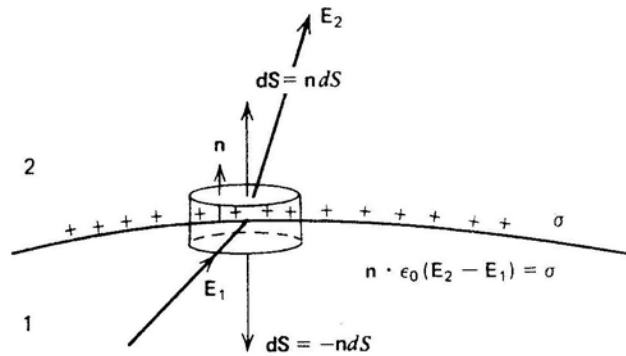
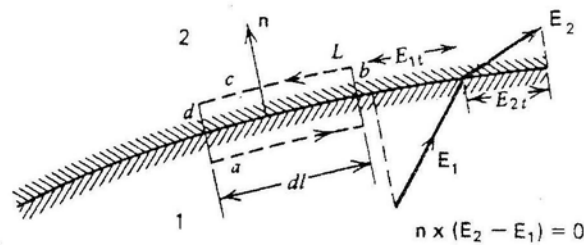


Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\epsilon_0 \mathbf{E}$ is discontinuous in the surface charge density.

$$\oint_S \bar{D} \cdot d\bar{a} = \int_S \sigma_{sf} dS \Rightarrow (D_{2n} - D_{1n}) dS = \sigma_{sf} dS$$

$$\bar{n} \cdot [(\bar{D}_2 - \bar{D}_1)] = \sigma_{sf}$$

2. Continuity of Tangential \bar{E}



(a)

Figure 3-12 (a) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary.

$$\oint_C \vec{E} \cdot d\vec{s} = (E_{1t} - E_{2t}) dl = 0 \Rightarrow E_{1t} - E_{2t} = 0$$

$$\vec{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

Equivalent to $\Phi_1 = \Phi_2$ along boundary

V. Solutions to Laplace's Equation in Cartesian Coordinates, $\Phi(x, y)$

$$\nabla^2 \Phi(x, y) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

1. Try product solution: $\Phi(x, y) = X(x) Y(y)$

$$Y(y) \frac{d^2 X(x)}{dx^2} + X(x) \frac{d^2 Y(y)}{dy^2} = 0$$

Multiply through by $\frac{1}{XY}$:

$$\underbrace{\frac{1}{X} \frac{d^2 X}{dx^2}}_{\text{only a function of } x} = - \underbrace{\frac{1}{Y} \frac{d^2 Y}{dy^2}}_{\text{only a function of } y} = \underbrace{-k^2}_{k=\text{separation constant}}$$

only a function of x only a function of y

$$\frac{d^2 X}{dx^2} = -k^2 X \quad ; \quad \frac{d^2 Y}{dy^2} = k^2 Y$$

2. Zero Separation Constant Solutions: $k=0$

$$\frac{d^2 X}{dx^2} = 0 \Rightarrow X = a_1 x + b_1$$

$$\frac{d^2 Y}{dy^2} = 0 \Rightarrow Y = c_1 y + d_1$$

$$\Phi(x, y) = XY = a_2 + b_2 x + c_2 y + d_2 xy$$

3. Non-Zero Separation Constant Solutions: $k \neq 0$

$$\frac{d^2 X}{dx^2} + k^2 X = 0 \Rightarrow X = A_1 \sin kx + A_2 \cos kx$$

$$\frac{d^2 Y}{dy^2} - k^2 Y = 0 \Rightarrow Y = B_1 e^{ky} + B_2 e^{-ky}$$

$$= C_1 \sinh ky + D_1 \cosh ky$$

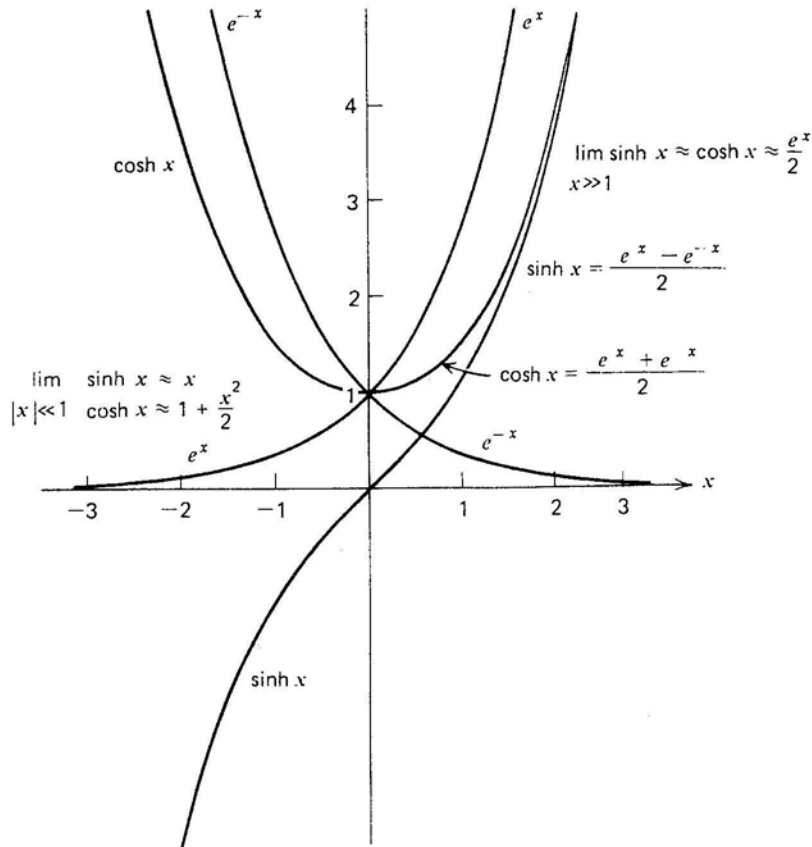
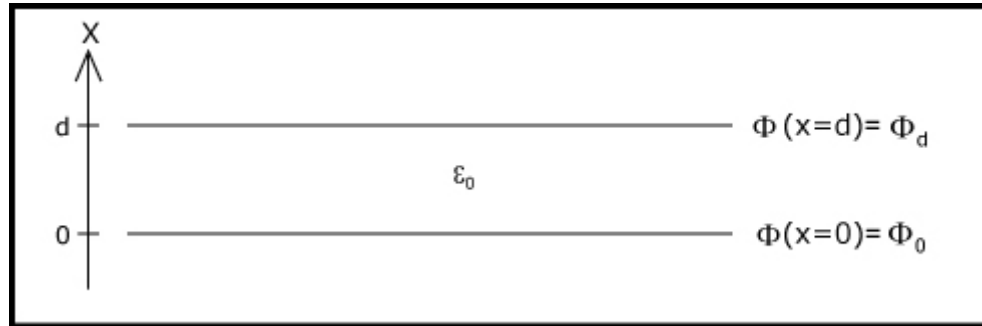


Figure 4-3 The exponential and hyperbolic functions for positive and negative arguments.

$$\Phi(x, y) = X(x)Y(y) = D_1 \sin kx e^{ky} + D_2 \sin kx e^{-ky} + D_3 \cos kx e^{ky} + D_4 \cos kx e^{-ky}$$

$$= E_1 \sin kx \sinh ky + E_2 \sin kx \cosh ky + E_3 \cos kx \sinh ky + E_4 \cos kx \cosh ky$$

4. Parallel Plate Electrodes



Neglecting end effects, $\Phi(x)$. Boundary conditions are:

$$\Phi(x=0) = \Phi_0, \quad \Phi(x=d) = \Phi_d$$

Try zero separation constant solution:

$$\Phi(x) = a_1 x + b_1$$

$$\Phi(x=0) = \Phi_0 = b_1$$

$$\Phi(x=d) = \Phi_d = a_1 d + b_1 \Rightarrow a_1 = \frac{\Phi_d - \Phi_0}{d}$$

$$\Phi(x) = \frac{\Phi_d - \Phi_0}{d} x + \Phi_0$$

$$E_x = -\frac{d\Phi}{dx} = \frac{\Phi_0 - \Phi_d}{d} \quad (\text{Electric field is uniform and equal to potential difference divided by spacing})$$

5. Hyperbolic Electrode Boundary Conditions

$$\Phi(xy = ab) = V_0$$

$$\Phi(x=0, y) = 0$$

$$\Phi(x, y=0) = 0$$

$$\Phi(x, y) = V_0 xy/(ab)$$

$$\begin{aligned} \vec{E} &= -\nabla\Phi = -\frac{\partial\Phi}{\partial x} \vec{i}_x - \frac{\partial\Phi}{\partial y} \vec{i}_y \\ &= -\frac{V_0}{ab} \left[y \vec{i}_x + x \vec{i}_y \right] \end{aligned}$$

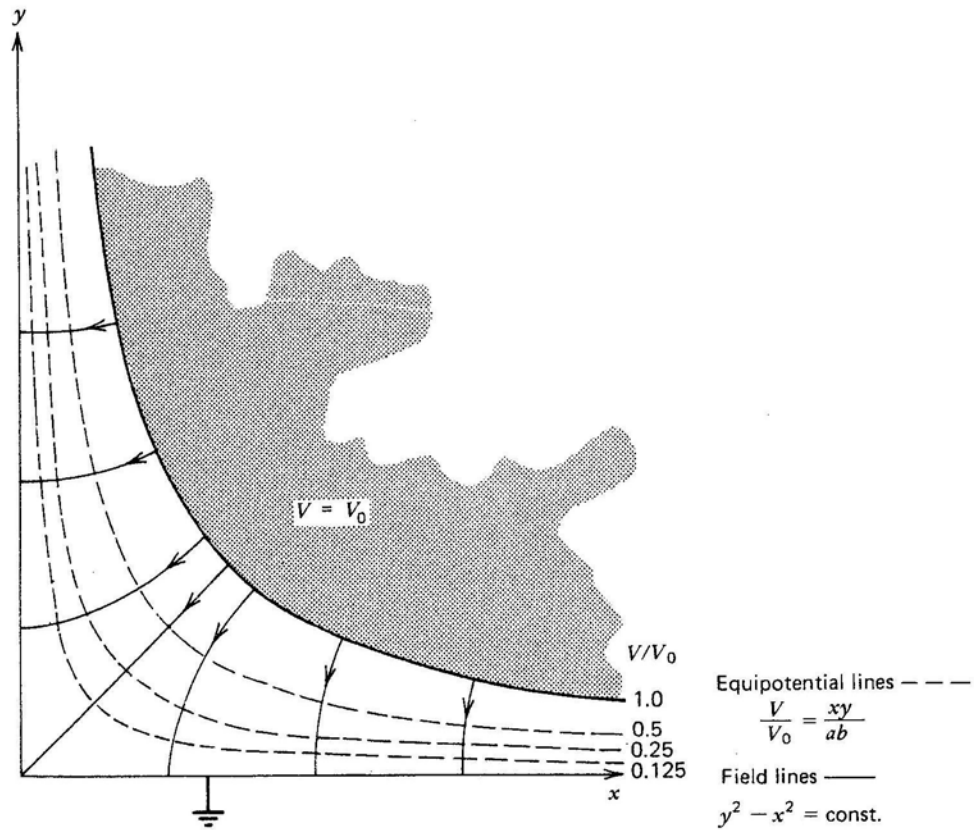


Figure 4-1 The equipotential and field lines for a hyperbolicly shaped electrode at potential V_0 above a right-angle conducting corner are orthogonal hyperbolas.

Electric field lines:

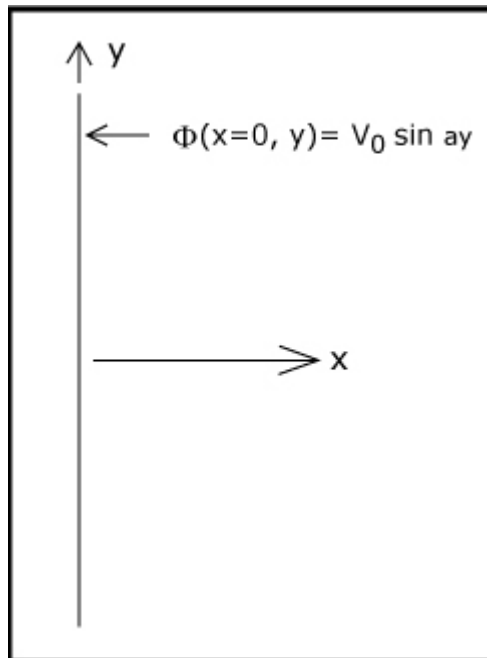
$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y}$$

$$ydy = xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 = x^2 + y_0^2 - x_0^2 \text{ (field line passes through } (x_0, y_0))$$

6. Spatially Periodic Potential Sheet



$$\Phi(x, y) = \begin{cases} V_0 \sin ay e^{-ax} & x \geq 0 \\ V_0 \sin ay e^{+ax} & x \leq 0 \end{cases}$$

$$\vec{E} = -\nabla\Phi(x, y) = -\left[\frac{\partial\Phi}{\partial x} \vec{i}_x + \frac{\partial\Phi}{\partial y} \vec{i}_y \right]$$

$$= \begin{cases} -V_0 a e^{-ax} \left[\cos ay \vec{i}_y - \sin ay \vec{i}_x \right] & x > 0 \\ -V_0 a e^{+ax} \left[\cos ay \vec{i}_y + \sin ay \vec{i}_x \right] & x < 0 \end{cases}$$

$$\sigma_s(x=0) = \epsilon_0 [E_x(x=0_+) - E_x(x=0_-)] = 2\epsilon_0 V_0 a \sin ay$$

Electric Field Lines:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \begin{cases} -\cot ay & x > 0 \\ +\cot ay & x < 0 \end{cases}$$

$$x > 0 \quad \cos ay e^{-ax} = \text{constant}$$

$$x < 0 \quad \cos ay e^{+ax} = \text{constant}$$

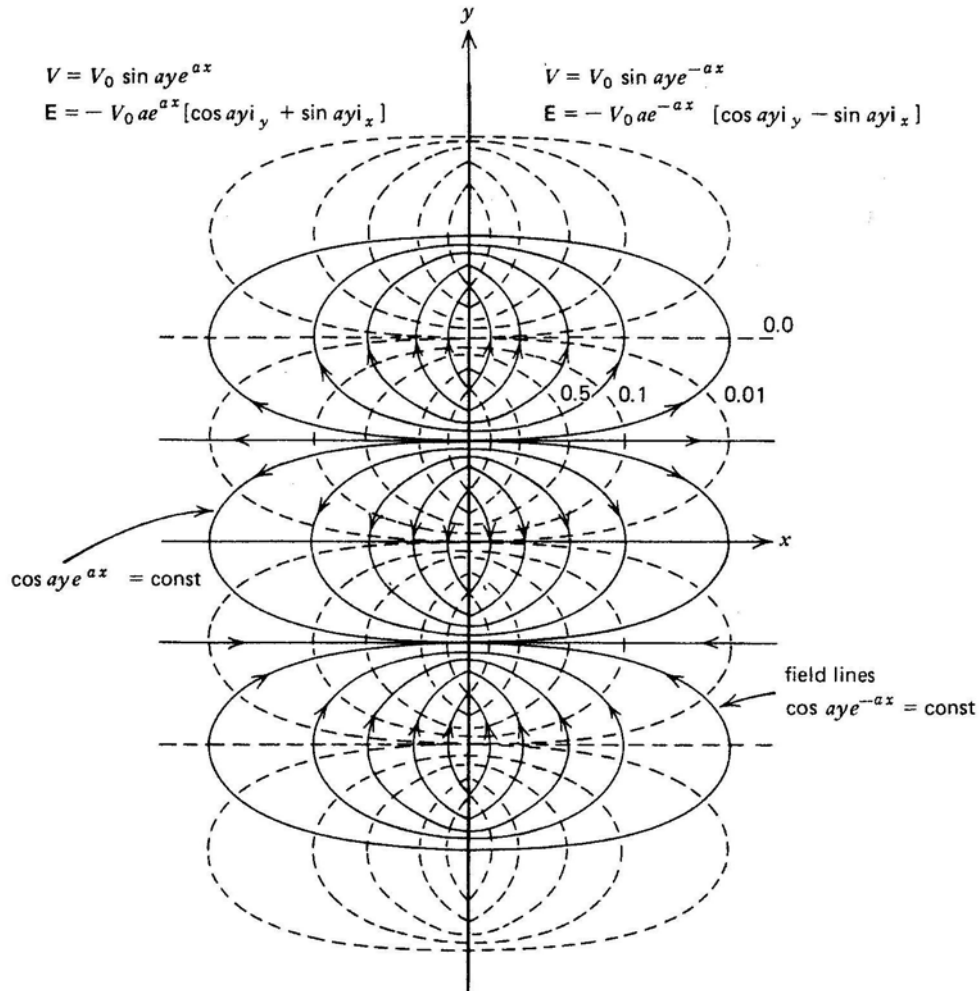


Figure 4-4 The potential and electric field decay away from an infinite sheet with imposed spatially periodic voltage. The field lines emanate from positive surface charge on the sheet and terminate on negative surface charge.