6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2006
Quiz 1 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 1

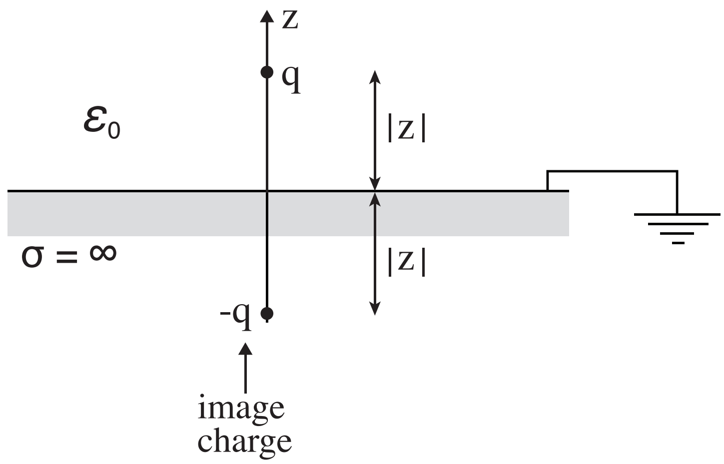

Figure 1: Charge q of a mass m above a perfectly conducting ground plane and its image charge $-q$. (Image by MIT OpenCourseWare.)

A

<u>Question:</u> What is the velocity of the charge as a function of position z?

<u>Solution:</u> Force = ma

$$\Rightarrow m\frac{d\overline{v}}{dt} = \overline{F}_{\text{due to image charge}}$$

$$\Rightarrow m\frac{d\overline{v}}{dt} = \frac{q(-q)}{4\pi\varepsilon_0(2z)^2}\overline{i}_z \Rightarrow m\frac{dv_z}{dt} = -\frac{q^2}{4\pi\varepsilon_0 4z^2}$$

$$\Rightarrow \frac{dv_z}{dt} = -\frac{q^2}{16\pi\varepsilon_0 m z^2} \text{ Use of chain rule: } \frac{dv_z}{dt} = \frac{dv_z}{dz}\frac{dz}{dt} = \frac{dv_z}{dz}v_z = \frac{d}{dz}\left(\frac{v_z^2}{2}\right)$$

$$\Rightarrow \frac{d}{dz}\left(\frac{v_z^2}{2}\right) = -\frac{q^2}{16\pi\varepsilon_0 m z^2} \Rightarrow \int d\left(\frac{v_z^2}{2}\right) = \int -\frac{q^2}{16\pi\varepsilon_0 m z^2} dz$$

$$\Rightarrow v_z^2 = +\frac{q^2}{8\pi\varepsilon_0 m z} + C$$

Use I.C. $v_z(z = d) = 0$

$$\Rightarrow \boxed{C = -\frac{q^2}{8\pi\varepsilon_0 m d}}$$

$$v_z^2 = \frac{q^2}{8\pi\varepsilon_0 m}\left(\frac{1}{z} - \frac{1}{d}\right)$$

$$\Rightarrow \boxed{v_z(z) = -\sqrt{\frac{q^2}{8\pi\varepsilon_0 m}\left(\frac{1}{z} - \frac{1}{d}\right)}}$$

since particle is moving towards the conducting ground plane $\overline{v} = v_z \overline{i}_z$ (v_z has minus sign)

B

<u>Question:</u> How long does it take the charge to reach the $z = 0$ ground plane?

Hint:

$$\int \frac{dz}{\left[\frac{1}{z} - \frac{1}{d}\right]^{\frac{1}{2}}} = -\sqrt{zd(d-z)} + d^{\frac{3}{2}} \tan^{-1}\sqrt{\frac{z}{d-z}}$$

<u>Solution:</u>

$$v_z = \frac{dz}{dt} = -\sqrt{\frac{q^2}{8\pi\varepsilon_0 m}\left(\frac{1}{z} - \frac{1}{d}\right)}$$

$$\Rightarrow \int \frac{dz}{\sqrt{\frac{1}{z} - \frac{1}{d}}} = \int -\sqrt{\frac{q^2}{8\pi\varepsilon_0 m}} dt$$

$$-\sqrt{zd(d-z)} + d^{\frac{3}{2}} \tan^{-1}\sqrt{\frac{z}{d-z}} = -t\sqrt{\frac{q^2}{8\pi\varepsilon_0 m}} + \underbrace{C}_{\text{constant}}$$

We use I.C. to find constant $\Rightarrow$ at $t = 0$, $z(t = 0) = d$.

$$\Rightarrow d^{\frac{3}{2}} \underbrace{\tan^{-1}\sqrt{\frac{d}{0}}}_{\frac{\pi}{2}} = C \Rightarrow C = \frac{\pi}{2} d^{\frac{3}{2}}$$

plugging in $z = 0$ gives the time (T) it takes for the charge to reach the $z = 0$ ground plane.

$$\Rightarrow 0 + d^{\frac{3}{2}} \underbrace{\tan^{-1}\sqrt{\frac{0}{d}}}_{0} = -\sqrt{\frac{q^2}{8\pi\varepsilon_0 m}} T + \frac{\pi}{2} d^{\frac{3}{2}}$$

$$\Rightarrow T = \frac{\frac{\pi}{2} d^{\frac{3}{2}}}{\sqrt{\frac{q^2}{8\pi\varepsilon_0 m}}} = \sqrt{\frac{\pi^2}{4} d^3 \frac{8\pi\varepsilon_0 m}{q^2}} \Rightarrow \boxed{T = \sqrt{\frac{2\pi^3 d^3 \varepsilon_0 m}{q^2}}}$$

Problem 2

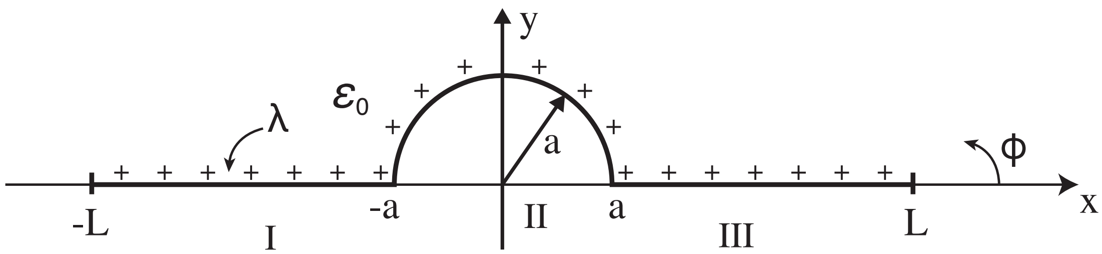

Figure 2: Uniformly distributed line charge λ. (Image by MIT OpenCourseWare.)

$$\Phi(\overline{r}) = \int_{l'} \frac{\lambda(\overline{r}')dl'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|} \qquad \overline{E}(\overline{r}) = \int_{l'} \frac{\lambda(\overline{r}')\overline{i}_{r'r} dl'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|^2}$$

A

Question: **Find the potential at the point** $(x = 0, y = 0, z = 0)$

$$\begin{aligned}
\Phi(x = 0, y = 0, z = 0) &= \int_{x=-L}^{-a} \frac{\lambda dx}{4\pi\varepsilon_0(-x)} + \int_{\phi=0}^{\pi} \frac{\lambda a d\phi}{4\pi\varepsilon_0 a} + \int_{x=a}^{L} \frac{\lambda dx}{4\pi\varepsilon_0(x)} \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left[-\ln x \Big|_{-L}^{-a} + \phi \Big|_0^{\pi} + \ln x \Big|_a^L \right] \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left[-\ln\frac{a}{L} + \pi + \ln\frac{L}{a} \right] = \boxed{\frac{\lambda}{4\pi\varepsilon_0}(\pi + 2\ln\frac{L}{a})}
\end{aligned}$$

B

Question: **Find the electric field (magnitude and direction) at** $(x = 0, y = 0, z = 0)$**.**

Solution:

$$\begin{aligned}\overline{E}(x=0,y=0,z=0) &= \int_{x=-L}^{-a} \frac{\lambda \bar{i}_x dx}{4\pi\varepsilon_0 x^2} + \int_{\phi=0}^{\pi} \frac{\lambda(-\bar{i}_r) a d\phi}{4\pi\varepsilon_0 a^2} + \int_{x=a}^{L} \frac{\lambda(-\bar{i}_x) dx}{4\pi\varepsilon_0 x^2} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{x}\Big|_{-L}^{-a} \bar{i}_x + \int_{\phi=0}^{\pi} \frac{-\cos\phi \bar{i}_x - \sin\phi \bar{i}_y}{a} d\phi + \frac{1}{x}\Big|_a^L \bar{i}_x \right] \\ &= \frac{\lambda}{4\pi\varepsilon_0}\left[\left(-\frac{1}{-a} + \frac{1}{-L} \right)\bar{i}_x + \frac{1}{a}\left(\underbrace{-\sin\phi}_{0}\Big|_0^{\pi} \bar{i}_x + \underbrace{\cos\phi}_{-2}\Big|_0^{\pi} \bar{i}_y \right) + \left(\frac{1}{L} - \frac{1}{a} \right)\bar{i}_x \right] \\ &= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{-2}{a} \right)\bar{i}_y \Rightarrow \boxed{\overline{E}(x=0,y=0,z=0) = -\frac{\lambda}{2\pi\varepsilon_0 a}\bar{i}_y}\end{aligned}$$

Problem 3

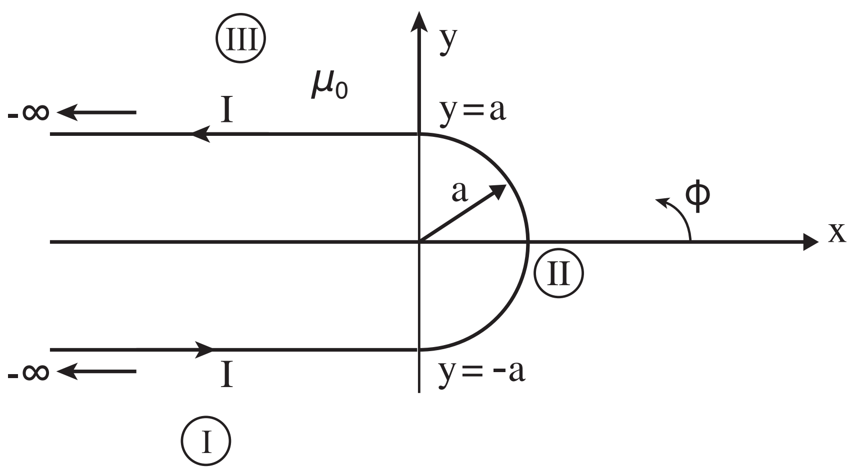

Figure 3: Current. (Image by MIT OpenCourseWare.)

Question: What is the magnetic field $\overline{H}$ at the point $(x=0, y=0, z=0)$?

Hint: One or more of the following indefinite integrals may be useful.

a. $\int \frac{dx}{[x^2+a^2]^{\frac{1}{2}}} = \ln\left[x + \sqrt{x^2+a^2}\right]$

b. $\int \frac{x dx}{[x^2+a^2]^{\frac{1}{2}}} = \left[x^2+a^2\right]^{\frac{1}{2}}$

c. $\int \frac{dx}{[x^2+a^2]} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

d. $\int \frac{dx}{[x^2+a^2]^{\frac{3}{2}}} = \frac{x}{a^2[x^2+a^2]^{\frac{1}{2}}}$

e. $\int \frac{x dx}{[x^2+a^2]^{\frac{3}{2}}} = -\frac{1}{[x^2+a^2]^{\frac{1}{2}}}$

Solution:

$$\overline{H}(\overline{r}) = \frac{1}{4\pi}\int \frac{\overline{I}dl' \times \overline{i}_{r'r}}{|\overline{r}-\overline{r}'|^2}$$

$$\overline{H}(x=0, y=0, z=0)$$

$$\begin{aligned}
&= \frac{1}{4\pi}\left\{\int_{x=-\infty}^{0} \frac{I\overline{i}_x dx \times (-x\overline{i}_x + a\overline{i}_y)}{(x^2+a^2)^{\frac{3}{2}}} + \int_{\phi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{I\overline{i}_\phi a d\phi \times (-\overline{i}_r)}{a^2} + \int_{x=-\infty}^{0} \frac{I(-\overline{i}_x)dx \times (-x\overline{i}_x - a\overline{i}_y)}{(x^2+a^2)^{\frac{3}{2}}}\right\} \\
&= \frac{I}{4\pi}\left\{\int_{x=-\infty}^{0} \frac{a\overline{i}_z dx}{(x^2+a^2)^{\frac{3}{2}}} + \int_{\phi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\overline{i}_z d\phi}{a} + \int_{x=-\infty}^{0} \frac{a\overline{i}_z dx}{(x^2+a^2)^{\frac{3}{2}}}\right\} \\
&= \frac{I}{4\pi}\left\{\frac{1}{a}\frac{x}{(x^2+a^2)^{\frac{1}{2}}}\Big|_{-\infty}^{0}\overline{i}_z + \frac{1}{a}\phi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\overline{i}_z + \frac{1}{a}\frac{x}{(x^2+a^2)^{\frac{3}{2}}}\Big|_{-\infty}^{0}\overline{i}_z\right\} \\
&= \frac{I}{4\pi a} i_z\left\{0-(-1)+\frac{\pi}{2}+\frac{\pi}{2}+0-(-1)\right\} \\
&= \boxed{\tfrac{I}{4\pi a}(2+\pi)\,\overline{i}_z}
\end{aligned}$$