6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn
Lecture 17: Transient Waves on Transmission Lines

I. Wave Equation (Loss Less)

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

$$\Rightarrow \frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \qquad c^2 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Solution: $v(z,t) = V_+\left(t-\frac{z}{c}\right) + V_-\left(t+\frac{z}{c}\right)$

Proof: Let $\alpha = t-\frac{z}{c} \Rightarrow \frac{\partial\alpha}{\partial t} = 1,\ \frac{\partial\alpha}{\partial z} = -\frac{1}{c}$ (Positive z directed waves)

Superposition: $v_+(z,t) = V_+\left(t-\frac{z}{c}\right) = V_+(\alpha)$

$$\frac{\partial v_+}{\partial t} = \frac{dv_+}{d\alpha}\frac{\partial\alpha}{\partial t} = \frac{dv_+}{d\alpha}$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 v_+}{d\alpha^2}\frac{\partial\alpha}{\partial t} = \frac{d^2 v_+}{d\alpha^2}$$

$$\frac{\partial v_+}{\partial z} = \frac{dv_+}{d\alpha}\frac{\partial\alpha}{\partial z} = -\frac{1}{c}\frac{dv_+}{d\alpha}$$

$$\frac{\partial^2 v_+}{\partial z^2} = -\frac{1}{c}\frac{d^2 v_+}{d\alpha^2}\frac{\partial\alpha}{\partial z} = +\frac{1}{c^2}\frac{d^2 v_+}{d\alpha^2}$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 v_+}{d\alpha^2} = c^2\frac{\partial^2 v}{\partial z^2} = c^2\left(\frac{1}{c^2}\frac{d^2 v_+}{d\alpha^2}\right) = \frac{d^2 v_+}{d\alpha^2}$$

Negative z directed waves: Let $\beta = t + \frac{z}{c} \Rightarrow \frac{\partial \beta}{\partial t} = 1,\ \frac{\partial \beta}{\partial z} = \frac{1}{c}$

$$\frac{\partial v_-}{\partial t} = \frac{dv_-}{d\beta} \frac{\partial \beta}{\partial t} = \frac{dv_-}{d\beta}$$

$$\frac{\partial^2 v_-}{\partial t^2} = \frac{d^2 v_-}{d\beta^2} \frac{\partial \beta}{\partial t} = \frac{d^2 v_-}{d\beta^2}$$

$$\frac{\partial v_-}{\partial z} = \frac{dv_-}{d\beta} \frac{\partial \beta}{\partial z} = \frac{1}{c} \frac{dv_-}{d\beta}$$

$$\frac{\partial^2 v_-}{\partial z^2} = \frac{1}{c} \frac{d^2 v_-}{d\beta^2} \frac{\partial \beta}{\partial z} = \frac{1}{c^2} \frac{d^2 v_-}{d\beta^2}$$

$$\frac{\partial^2 v_-}{\partial t^2} = \frac{d^2 v_-}{d\beta^2} = c^2 \frac{\partial^2 v_-}{\partial z^2} = c^2 \left(\frac{1}{c^2} \frac{d^2 v_-}{d\beta^2} \right) = \frac{d^2 v_-}{d\beta^2}$$

II. Solution for current $i(z,t)$

$$\left.\begin{aligned} \frac{\partial v}{\partial z} &= -L \frac{\partial i}{\partial t} \\ \frac{\partial i}{\partial z} &= -C \frac{\partial v}{\partial t} \end{aligned}\right\} \Rightarrow \frac{\partial^2 i}{\partial t^2} = c^2 \frac{\partial^2 i}{\partial z^2}$$

Solution: $i(z,t) = I_+\left(t - \frac{z}{c}\right) + I_-\left(t + \frac{z}{c}\right)$

$$v(z,t) = V_+\left(t - \frac{z}{c}\right) + V_-\left(t + \frac{z}{c}\right)$$

+z Solution: $\alpha = t - \frac{z}{c},\ \frac{\partial \alpha}{\partial t} = 1,\ \frac{\partial \alpha}{\partial z} = -\frac{1}{c}$

$$\frac{\partial v_+}{\partial z} = -L \frac{\partial i_+}{\partial t} \Rightarrow \frac{dv_+}{d\alpha} \frac{\partial \alpha}{\partial z} = -\frac{1}{c} \frac{dv_+}{d\alpha} = -L \frac{di_+}{d\alpha} \frac{\partial \alpha}{\partial t}$$

$$= -L \frac{di_+}{d\alpha}$$

$$\frac{dv_+}{d\alpha} = L\,c\,\frac{di_+}{d\alpha} = \frac{L}{\sqrt{LC}}\,\frac{di_+}{d\alpha} = \sqrt{\frac{L}{C}}\,\frac{di_+}{d\alpha}$$

$$v_+ = i_+ Z_0 \Rightarrow I_+\left(t-\frac{z}{c}\right) = Y_0 V_+\left(t-\frac{z}{c}\right)$$

$$Y_0 = \sqrt{\frac{C}{L}} = \frac{1}{Z_0}$$

-z Solution: $\beta = t+\frac{z}{c}$, $\frac{\partial\beta}{\partial t} = 1$, $\frac{\partial\beta}{\partial z} = \frac{1}{c}$

$$\frac{\partial v_-}{\partial z} = -L\frac{\partial i_-}{\partial t} \Rightarrow \frac{dv_-}{d\beta}\,\frac{\partial\beta}{\partial z} = \frac{1}{c}\,\frac{dv_-}{d\beta} = -L\,\frac{di_-}{d\beta}\,\frac{\partial\beta}{\partial t}$$

$$= -L\,\frac{di_-}{d\beta}$$

$$\frac{dv_-}{d\beta} = -Lc\,\frac{di_-}{d\beta} = \frac{-L}{\sqrt{LC}}\,\frac{di_-}{d\beta} = -\sqrt{\frac{L}{C}}\,\frac{di_-}{d\beta}$$

$$v_- = -Z_0\,i_- \Rightarrow I_-\left(t+\frac{z}{c}\right) = -Y_0\,V_-\left(t+\frac{z}{c}\right)$$

$$v(z,t) = V_+\left(t-\frac{z}{c}\right) + V_-\left(t+\frac{z}{c}\right)$$

$$i(z,t) = Y_0\left[V_+\left(t-\frac{z}{c}\right) - V_-\left(t+\frac{z}{c}\right)\right]$$

III. Transmission Line Transient Waves

A. Transients on Infinitely Long Transmission Lines

1. Initial Conditions

$$v(z, t=0) = 0 = V_+\left(-\frac{z}{c}\right) + V_-\left(\frac{z}{c}\right) = 0$$

$$i(z, t=0) = 0 = Y_0\left[V_+\left(-\frac{z}{c}\right) - V_-\left(\frac{z}{c}\right)\right] = 0$$

$$V_+\left(-\frac{z}{c}\right) = 0, \quad V_-\left(\frac{z}{c}\right) = 0$$

$$z > 0, \ \ t > 0 \ \Rightarrow t + \frac{z}{c} > 0 \ \Rightarrow V_-\left(t + \frac{z}{c}\right) = 0$$

$$t - \frac{z}{c} > 0 \quad \text{if} \quad t > \frac{z}{c} \quad \text{to allow} \ \ V_+\left(t - \frac{z}{c}\right) \neq 0$$

$$\text{With} \ \ V_-\left(t + \frac{z}{c}\right) = 0 \Rightarrow \ v(z,t) = V_+\left(t - \frac{z}{c}\right)$$

$$i(z,t) = Y_0 V_+\left(t - \frac{z}{c}\right) \quad \Rightarrow \frac{v(z,t)}{i(z,t)} = Z_0$$

2. Traveling Wave Solution with Source Resistance R_s

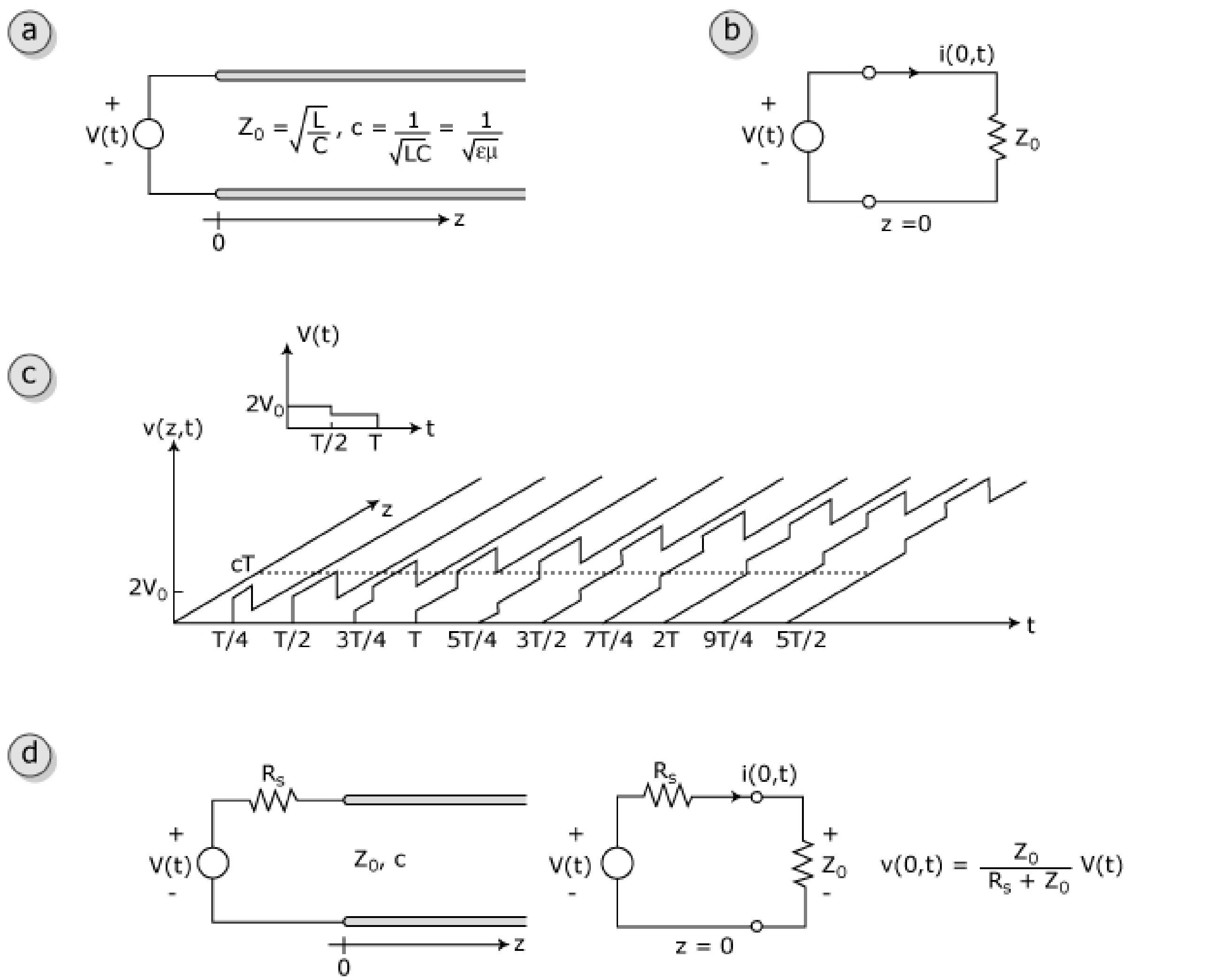
$$v(z=0,t) = V(t) = V_+(t)$$

$$v(z=0, \ t) = \frac{Z_0}{Z_0 + R_s} V(t) = V_+(t)$$

$$i(z=0,t) = Y_0 V_+(t) = \frac{V(t)}{R_s + Z_0}$$

$$v(z, t) = \frac{Z_0}{Z_0 + R_s} V\left(t - \frac{z}{c}\right)$$

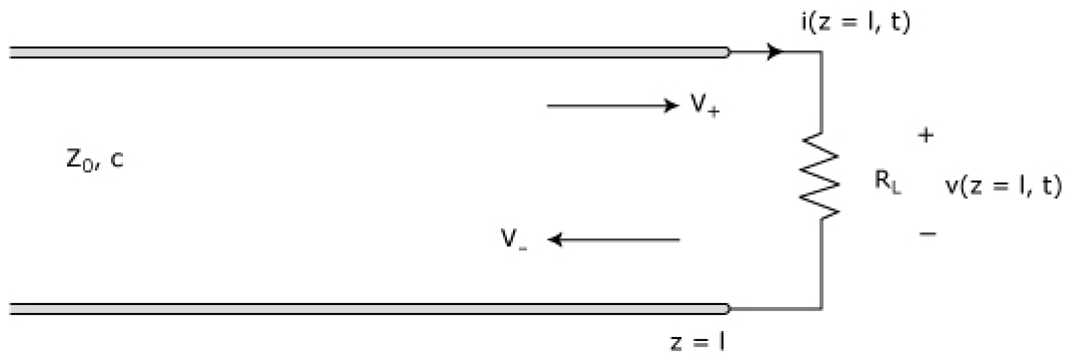
$$i(z,t) = \frac{V\left(t - \frac{z}{c}\right)}{R_s + Z_0}$$



(a) A semi-infinite transmission line excited by a voltage source at z = 0. (b) To the source, the transmission line looks like a resistor Z_0 equal to the characteristic impedance. (c) The spatial distribution of the voltage v(z,t) at various times for a staircase pulse of V(t). (d) If the voltage source is applied to the transmission line through a series resistance R_s the voltage across the line at z = 0 is given by the voltage divider relation.

B. Reflections from Resistive Terminations

1. Reflection Coefficient



$$v(z = l, t) = V_+ + V_- = i(z = l, t)R_L = R_L Y_0 [V_+ - V_-]$$
$$= \Gamma_L = \frac{V_-}{V_+} = \frac{R_L - Z_0}{R_L + Z_0}$$

A V_+ wave incident upon the end of a transmission line with a load resistor R_L is reflected as a V_- wave.

$$\text{at } z = l : v(l,t) = V_+(t - l/c) + V_-(t + l/c)$$

$$= i(l,t)R_L$$

$$= Y_0 R_L \left[V_+\left(t - \frac{l}{c}\right) - V_-(t + l/c)\right]$$

$$\Gamma_L = \frac{V_-(t + l/c)}{V_+(t - l/c)} = \frac{R_L - Z_0}{R_L + Z_0}$$

Special Cases

i) $R_L = Z_0 \Rightarrow \Gamma_L = 0$ (matched line)

ii) $R_L = 0 \Rightarrow \Gamma_L = -1$ (short circuited line)

If $R_L < Z_0$, $\quad -1 < \Gamma_L < 0$

iii) $R_L = \infty \Rightarrow \Gamma_L = +1$ (open circuited line)

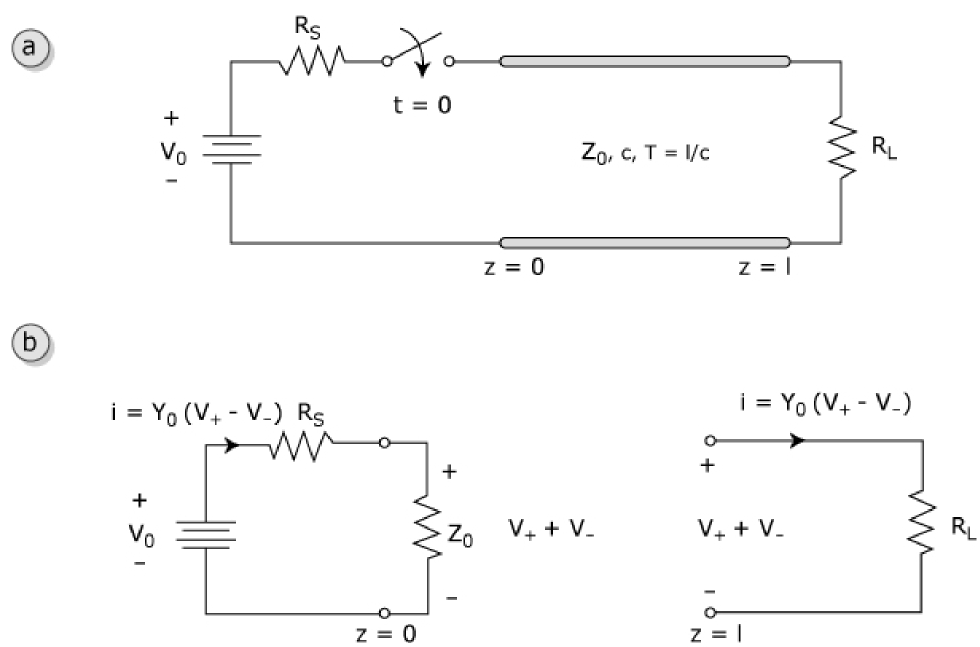
If $R_L > Z_0$, $\quad 0 < \Gamma_L < 1$

2. Step Voltage

At $z = 0 : v(z = 0, t) + i(0, t) R_S = V_0$

$$V_+(z = 0, t) + V_-(z = 0, t) + Y_0 R_s \left[V_+(z = 0, t) - V_-(z = 0, t)\right] = V_0$$

$$V_+(z = 0, t) = \Gamma_s V_-(z = 0, t) + \frac{Z_0 V_0}{Z_0 + R_s}, \quad \Gamma_s = \frac{R_s - Z_o}{R_s + Z_0}$$



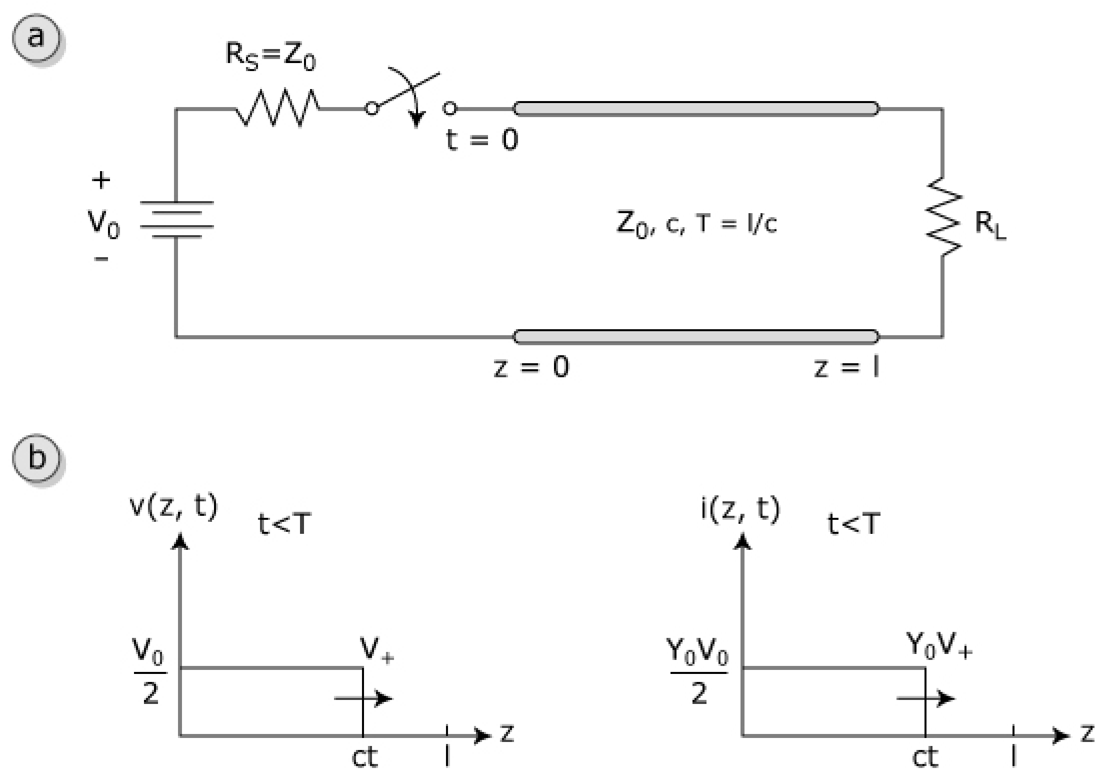
$$V_+ = \underbrace{\frac{Z_0}{R_S + Z_0}}_{\Gamma_0} V_0 + \underbrace{\frac{R_S - Z_0}{R_S + Z_0}}_{\Gamma_S} V_- \qquad V_- = \underbrace{\frac{R_L - Z_0}{R_L + Z_0}}_{\Gamma_L} V_+$$

(a) A dc voltage V_0 is switched onto a resistively loaded transmission line through a source resistance R_S
(b) The equivalent circuits at z = 0 and z = l allow us to calculate the reflected voltage wave amplitudes in terms of the incident waves.

a. Matched Line: $R_L = Z_0$, $\Gamma_L = 0$; $R_s = Z_0$, $\Gamma_s = 0$

$$\Gamma_L = 0 \Rightarrow V_-\left(t + \frac{z}{c}\right) = 0$$

$$V_+(z = 0, t) = \frac{V_0}{2}, \text{ In steady state after time } T = \frac{l}{c}$$

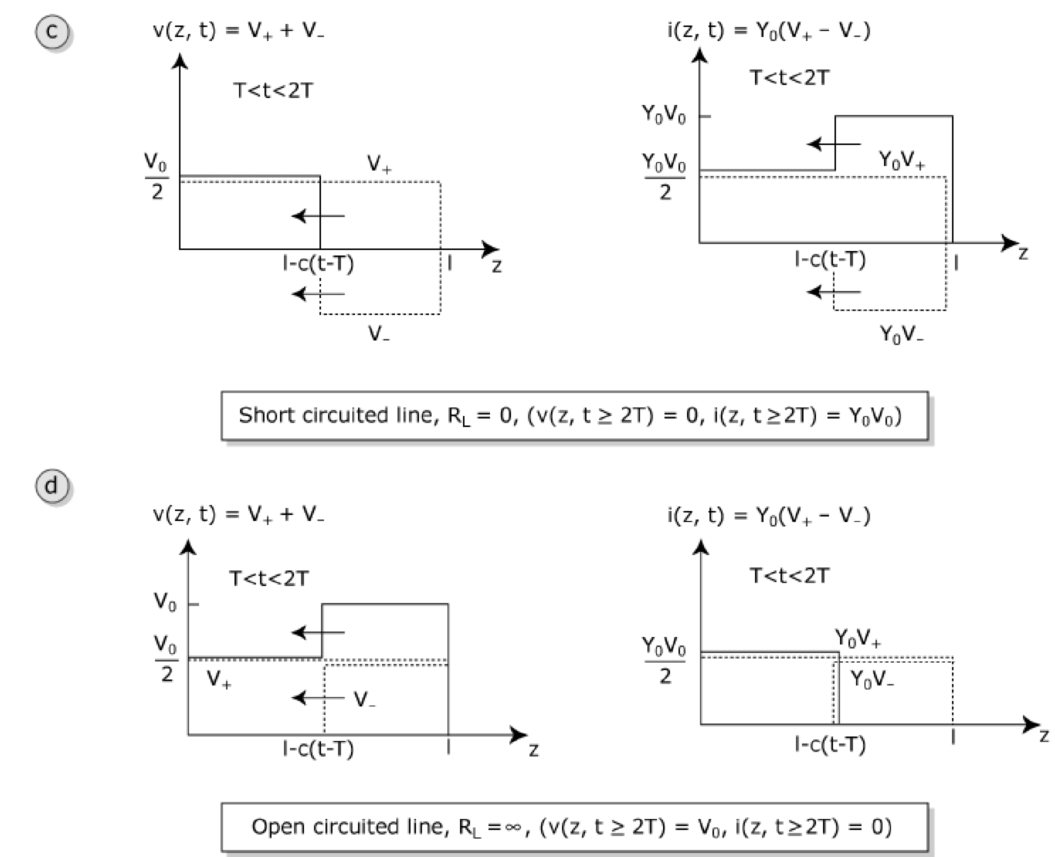


(a) A dc voltage is switched onto a transmission line with load resistance R_L through a source resistance $R_S = Z_0$ matched to the line. (b) Regardless of the load resistance, half the source voltage propagates down the line towards the load. If the load is also matched to the line ($R_L = Z_0$), there are no reflections and the steady state of $v(z, t \geq T) = V_0/2$, $i(z, t \geq T) = Y_0V_0/2$ is reached for $t \geq T$.

b. Short circuited line: $R_L = 0,\quad \Gamma_L = -1, R_s = Z_{0,}\quad \Gamma_s = 0$

$\Gamma_L = -1 \Rightarrow V_+ = -V_-$. When $V_+\left(t - \frac{z}{c}\right)$ and $V_-\left(t + \frac{z}{c}\right)$ overlap in space, $v(z,t) = 0$. For $t \geq 2T = 2l/c$, $v(z,t) = 0$,

$$i(z,t) = V_0/Z_0$$



(c) If the line is short circuited ($R_L = 0$), then $\Gamma_L = -1$ so that the V_+ and V_- waves cancel for the voltage but add for the current wherever they overlap in space. Since the source end is matched, no further reflections arise at $z = 0$ so the steady state is reached for $t \geq 2T$. (d) If the line is open circuited ($R_L = \infty$) so that $\Gamma_L = +1$, the V_+ and V_- waves add for the voltage but cancel for the current.

c. Open circuited line: $R_L = \infty, \Gamma_L = +1, R_s = Z_0, \Gamma_s = 0$

$$\Gamma_L = +1 \Rightarrow V_+ = +V_-$$

For $t \geq 2T = 2l/c,\quad v(z,t) = V_0,\quad i(z,t) = 0$

3. Approach to the dc Steady State (neither end matched)

$$z = 0: \quad V_+(t) = \Gamma_0 V_0 + \Gamma_s V_-(t), \; \Gamma_0 = \frac{Z_0}{R_s + Z_0}, \; \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0}$$

$$z = l: \quad V_-\left(t + \frac{l}{c}\right) = \Gamma_L V_+\left(t - \frac{l}{c}\right), \; \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

at $z = l$:

$$V_{+n} = \Gamma_0 V_0 + \Gamma_s V_{-(n-1)}$$

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)}$$

$$V_{+n} = \Gamma_0 V_0 + \Gamma_s \Gamma_L V_{+(n-1)} \qquad \Rightarrow V_{+n} - \Gamma_s \Gamma_L V_{+(n-1)} = \Gamma_0 V_0$$

Particular Solution: $V_{+n} = \text{constant}$

$$\text{Constant}\left(1 - \Gamma_s \Gamma_L\right) = \Gamma_0 V_0 \Rightarrow \text{Constant} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L}$$

Homogeneous Solution: $V_{+n} - \Gamma_s \Gamma_L V_{+(n-1)} = 0$

Try a solution of the form: $V_{+n} = A\lambda^n$

$$A\left(\lambda^n - \Gamma_s \Gamma_L \lambda^{n-1}\right) \Rightarrow \lambda = \Gamma_s \Gamma_L$$

$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} + A\left(\Gamma_s \Gamma_L\right)^n$$

Initial Condition:

$$V_{+1} = \Gamma_0 V_0 = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} + A\left(\Gamma_s \Gamma_L\right) \Rightarrow A = -\frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L}$$

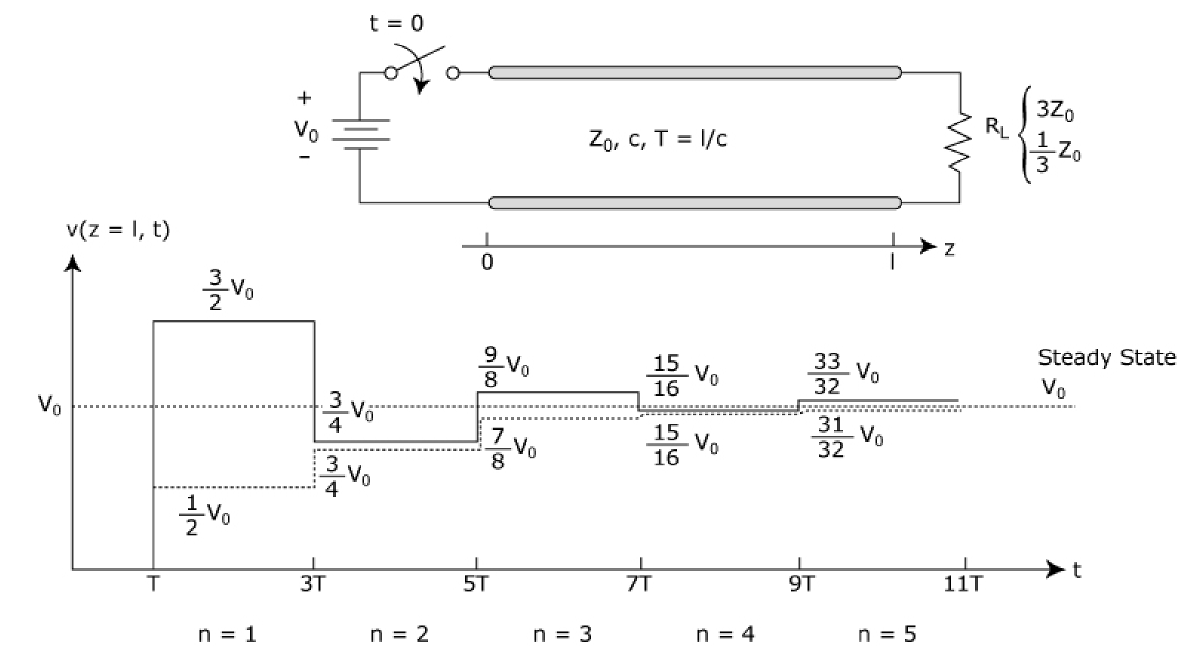
$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L}\left[1 - \left(\Gamma_s \Gamma_L\right)^n\right]$$

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)} \Rightarrow V_{-n} = \Gamma_L V_{+n}$$

$$V_n = V_{+n} + V_{-n} = V_{+n}\left(1+\Gamma_L\right) = \frac{V_0\left(1+\Gamma_L\right)\Gamma_0 V_0}{1-\Gamma_s\Gamma_L}\left[1-\left(\Gamma_s\Gamma_L\right)^n\right]$$

$$= \frac{R_L}{R_L+R_s} V_0\left[1-\left(\Gamma_s\Gamma_L\right)^n\right]$$

$$\lim_{n\to\infty} V_n = \frac{R_L}{R_L+R_s} V_0$$



The load voltage as a function of time when $R_S = 0$ and $R_L = 3Z_0$ so that $\Gamma_S\Gamma_L = \frac{-1}{2}$ (solid) and with $R_L = \frac{1}{3}Z_0$ so that $\Gamma_S\Gamma_L = \frac{1}{2}$ (dashed). The dc steady state is the same as if the transmission line were considered a pair of perfectly conducting wires in a circuit.

$$I_n = Y_0\left[V_{+n} - V_{-n}\right] = Y_0\left(1-\Gamma_L\right)V_{+n} = \frac{Y_0\left(1-\Gamma_L\right)\Gamma_0 V_0\left[1-\left(\Gamma_s\Gamma_L\right)^n\right]}{1-\Gamma_s\Gamma_L}$$

$$= V_0\left[1-\left(\Gamma_s\Gamma_L\right)^n\right] \Big/ \left(R_L+R_s\right)$$

a. Special Case: $R_s = 0, R_L = 3Z_0$

$$\Gamma_s = -1, \ \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{2}{4} = \frac{1}{2} \Rightarrow \Gamma_s \Gamma_L = -\frac{1}{2}$$

$$z = l \qquad V_n = V_0\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

$$I_n = \frac{V_0}{3Z_0}\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

b. Special Case: $R_s = 0, \ R_L = \frac{1}{3}Z_0$

$$\Gamma_s = -1, \ \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{\frac{-2}{3}}{\frac{4}{3}} = -\frac{1}{2} \Rightarrow \Gamma_s \Gamma_L = +\frac{1}{2}$$

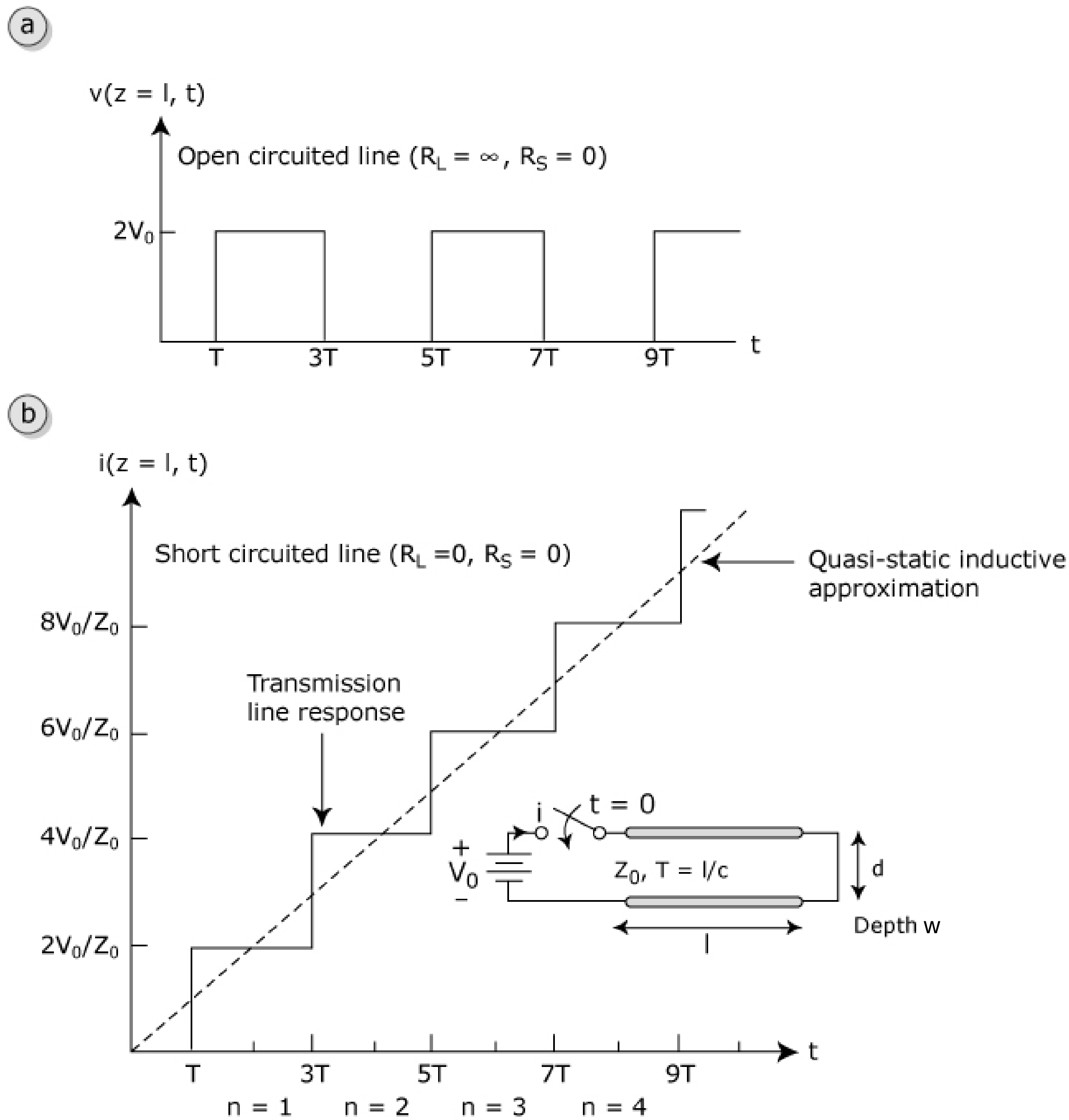
$$z = l \qquad V_n = V_0\left[1 - \left(\frac{1}{2}\right)^n\right]$$

c. Special Case: $R_s = 0, \ R_L = \infty$ (open circuit)

$$\Gamma_s \Gamma_L = -1$$

$$V_n = \frac{R_L}{R_s + R_L} V_0\left[1 - \left(\Gamma_s \Gamma_L\right)^n\right] = V_0\left(1 - (-1)^n\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ 2V_0 & n \text{ odd} \end{cases}$$



The (a) open circuit voltage and (b) short circuit current at the z = l end of the transmission line for $R_S = 0$. No dc steady state is reached because the system is lossless. If the short circuited transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.

d. Special Case: $R_s = 0,\ R_L = 0$ (Short circuit)

$$\Gamma_s \Gamma_L = +1$$

$$I_n = \frac{V_0}{R_L + R_S}\left[1 - \left(\Gamma_S \Gamma_L\right)^n\right] \text{ Indeterminate}$$

$$\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{\frac{R_S}{Z_0} - 1}{\frac{R_S}{Z_0} + 1} \approx -\left(1 - \frac{R_S}{Z_0}\right)^2 \approx -\left(1 - \frac{2R_S}{Z_0}\right)$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \approx -\left(1 - \frac{2R_L}{Z_0}\right)$$

$$I_n = \frac{V_0}{R_L + R_S}\left[1 - \left[\left(1 - \frac{2R_L}{Z_0}\right)\left(1 - \frac{2R_S}{Z_0}\right)\right]^n\right]$$

$$\approx \frac{V_0}{R_L + R_S}\left[1 - \left(1 - \frac{2\left(R_L + R_S\right)}{Z_0}\right)^n\right]$$

$$\approx \frac{V_0}{R_L + R_S}\left[1 - 1 + 2n\ \frac{\left(R_L + R_S\right)}{Z_0}\right]$$

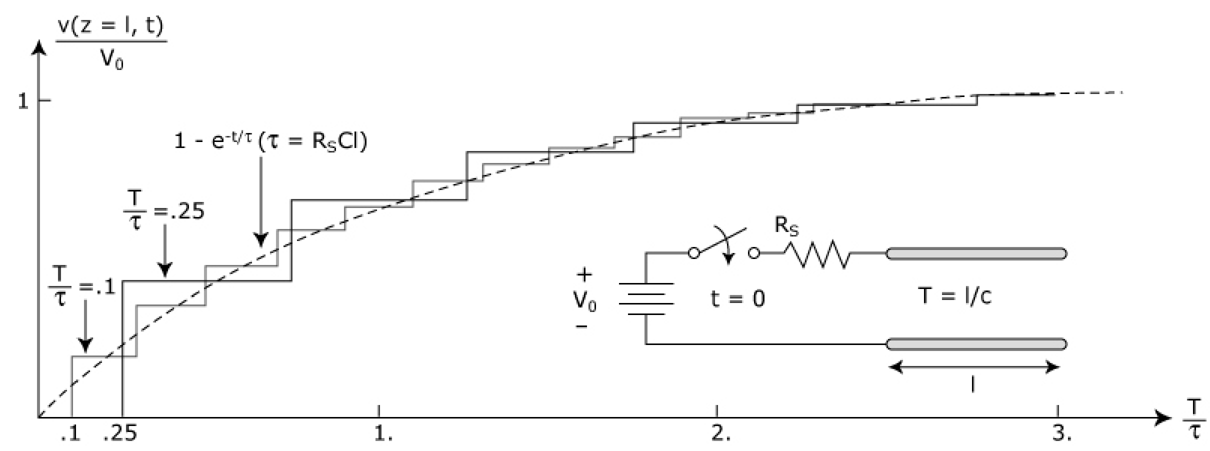
$$= \frac{V_0 2n}{Z_0}$$

Approximates Inductor

$$V_0 = \left(L l\right)\frac{di}{dt} \Rightarrow i = \frac{V_0}{L l} t$$

e. Special Case: $R_L = \infty$ (Open Circuit), R_s finite

$$\Gamma_L = 1 \Rightarrow V_n = V_0\left[1 - \Gamma_S^{\,n}\right]$$



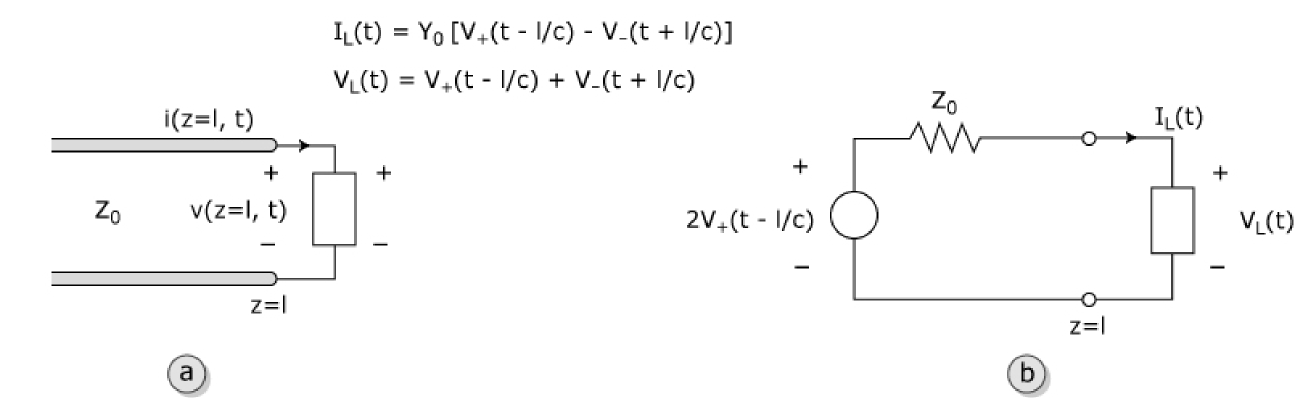
The open circuit voltage at z=l for a step voltage applied at t=0 through a source resistance R_S for various values of T/τ, which is the ratio of propagation time T=l/c to quasi-static charging time $\tau=R_SCl$. The dashed curve shows the exponential rise obtained by a circuit analysis assuming the open circuited transmission line is a capacitor.

Approximate transmission line as capacitor being charged though resistor R_S

$$v(t) = V_0\left(1 - e^{-t/t}\right)$$

$$T = R_S\,Cl$$

C. Reflections from Arbitrary Terminations



A transmission line with an (a) arbitrary load at the z=l end can be analyzed from the equivalent circuit in (b). Since V_+ is known, calculation of the load current or voltage yields the reflected wave V_-.

$$v(z=l,t) = V_L(t) = V_+\left(t-\frac{l}{c}\right) + V_-\left(t+\frac{l}{c}\right)$$

$$i(z=l,t) = I_L(t) = Y_0\left[V_+\left(t-\frac{l}{c}\right) - V-\left(t+\frac{l}{c}\right)\right]$$

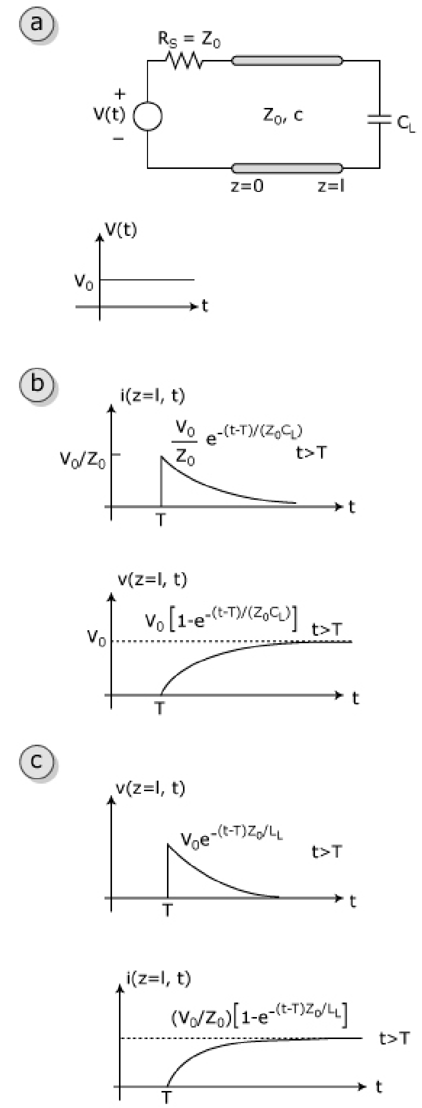
Eliminate $V_-\left(t+\frac{l}{c}\right) \Rightarrow 2V_+\left(t-\frac{l}{c}\right) = V_L(t) + I_L(t)Z_0$

$$V_-\left(t+\frac{l}{c}\right) = V_L(t) - V_+\left(t-\frac{l}{c}\right)$$

a. Capacitor C_L at $z=l$, $R_S = Z_0 \Rightarrow V_+ = \frac{V_0}{2}$

$$t > T \qquad V_L(t) = v_c(t),\ I_L(t) = C_L\frac{dv_c}{dt}$$

$$Z_0 C_L \frac{dv_c}{dt} + v_c = 2V_+ = V_0 \qquad t > T$$



(a) A step voltage is applied to transmission lines loaded at z=l with a capacitor C_L or inductor L_L. The load voltage and current are calculated from the (b) resistive-capacitive or (c) resistive-inductive equivalent circuits at z = l to yield exponential waveforms with respective time constants $\tau = Z_0C_L$ and $\tau = L_L/Z_0$ as the solutions approach the dc steady state. The waveforms begin after the initial V_+ wave arrives at z = l after a time T = l/c. There are no further reflections as the source end is matched.

$$v_c(t) = V_0\left[1 - e^{-(t-T)/(Z_0C_L)}\right] \quad t > T$$

$$T = l/c$$

$$V_- = v_c(t) - V_+$$

$$= \frac{-V_0}{2} + V_0\left[l - e^{-(t-T)/(Z_0C_L)}\right]$$

$$= \frac{V_0}{2} - V_0 e^{-(t-T)/(Z_0 C_L)}$$

$$i_c = C_L \frac{dv_c}{dt} = \frac{V_0}{Z_0} e^{-(t-T)/(Z_0 C_L)} \qquad t > T$$

b. Inductor L_L at $z = l$

$$L_L \frac{di_L}{dt} + i_L Z_0 = 2V_+ = V_0 \qquad t > T$$

$$i_L = \frac{V_0}{Z_0}\left(1 - e^{-(t-T)Z_0/L_L}\right) \qquad t > T$$

$$v_L = L_L \frac{di_L}{dt} = V_0 e^{-(t-T)Z_0/L_L} \qquad t > T$$

IV. Linear Constant Coefficient Difference Equations

$$D_n = \det \underbrace{\begin{bmatrix} a & c & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ c & a & c & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & c & a & c & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & 0 & c & a & c & 0 & 0 & 0 & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \end{bmatrix}}_{n} \; n$$

$$= a \begin{array}{cc} \xleftrightarrow{\qquad\qquad n-1 \qquad\qquad} & \\ \begin{bmatrix} a & c & 0 & 0 & 0 & 0 & - & - & - & - & - & - \\ c & a & c & 0 & 0 & 0 & - & - & - & - & - & - \\ 0 & c & a & c & 0 & 0 & - & - & - & - & - & - \\ 0 & 0 & c & a & c & 0 & - & - & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & \end{bmatrix} & \updownarrow n-1 \end{array}$$

$$-c^2 \begin{array}{cc} \xleftrightarrow{\qquad\qquad n-2 \qquad\qquad} & \\ \begin{bmatrix} a & c & 0 & 0 & 0 & 0 & - & - & - & - & - & - \\ c & a & c & 0 & 0 & 0 & - & - & - & - & - & - \\ 0 & c & a & c & 0 & 0 & - & - & - & - & - & - \\ 0 & 0 & c & a & c & 0 & - & - & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & \end{bmatrix} & \updownarrow n-2 \end{array}$$

$$D_n = a D_{n-1} - c^2 D_{n-2}$$

$$D_n - a D_{n-1} + c^2 D_{n-2} = 0$$

$$D_n = A\lambda^n$$

$$A\left[\lambda^n - a\lambda^{n-1} + c^2 \lambda^{n-2}\right] = 0 \Rightarrow A \lambda^{n-2}\left[\lambda^2 - a\lambda + c^2\right] = 0$$

$$\lambda^2 - a\lambda + c^2 = 0 \Rightarrow \lambda = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 - c^2\right]^{1/2}$$

$$= c\left[\frac{a}{2c} \pm \left[\left(\frac{a}{2c}\right)^2 - 1\right]^{1/2}\right]$$

Let $\frac{a}{2c} = \cos\theta$

$$\lambda = c\left[\cos\theta \pm \left[\cos^2\theta - 1\right]^{1/2}\right]$$

$$= c\left[\cos\theta \pm \left[-\sin^2\theta\right]^{1/2}\right]$$

$$= c\left[\cos\theta \pm j\sin\theta\right]$$

$$= c\,e^{\pm j\theta}$$

$$D_n = A_1\left(c\,e^{j\theta}\right)^n + A_2\left(c\,e^{-j\theta}\right)^n$$

$$= c^n\left[A_1 e^{jn\theta} + A_2 e^{-jn\theta}\right]$$

$$D_1 = a = c\left[A_1 e^{j\theta} + A_2 e^{-j\theta}\right]$$

$$D_2 = a^2 - c^2 = c^2\left[A_1 e^{j2\theta} + A_2 e^{-j2\theta}\right]$$

$$A_1 e^{j\theta} + A_2 e^{-j\theta} = \frac{a}{c}$$

$$\Rightarrow A_1 = \frac{e^{j\theta}}{2j\sin\theta}$$

$$A_2 = \frac{-e^{-j\theta}}{2j\sin\theta}$$

$$A_1 e^{j2\theta} + A_2 e^{-j2\theta} = \left(\frac{a}{c}\right)^2 - 1$$

$$D_n = \frac{c^n}{2j\sin\theta}\left[e^{j(n+1)\theta} - e^{-j(n+1)\theta}\right] = \frac{c^n \sin\left[(n+1)\theta\right]}{\sin\theta}$$

Check: $D_1 = c\frac{\sin 2\theta}{\sin\theta} = 2\cos\theta = \cancel{2c}\left(\frac{a}{\cancel{2c}}\right) = a$

$$D_2 = c^2\frac{\sin 3\theta}{\sin\theta} = c^2\left[4\cos^2\theta - 1\right] = c^2\left[4\left(\frac{a}{2c}\right)^2 - 1\right] = a^2 - c^2$$