Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

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Subject Organization and Technical Overview.

Subject Organization

Welcome to 6.453, *Quantum Optical Communication*. It is one of a collection of MIT classes that deals with aspects of an emerging field known as quantum information science. As you can divine from its title, 6.453 is about quantum *communication*, rather than quantum *computation*, although both of these topics fall under the general rubric of quantum information science. Moreover, 6.453 is far from being an entirely abstract presentation of quantum communication, although such a development is indeed possible, but is instead intimately tied to quantum optics. Finally, 6.453 does not presume a deep background in quantum mechanics or optics, such as would be obtained from one or more semesters of study in the Physics Department, but instead teaches all the basic quantum mechanics that is needed and does not require any electromagnetics knowledge beyond the plane-wave solutions to Maxwell's equations in a source-free region of empty space.

The preceding paragraph characterizes 6.453 as an outgrowth of quantum optics, i.e., the marriage of quantum mechanics and optics. An alternative, and more informative, way to look at 6.453 is as an outgrowth of communications and especially communication *theory*. This should be clear from its prerequisites being 6.011 and 18.06, which indicate that 6.453 will build on knowledge of signals and systems, probability, and linear algebra. In particular, we will rely on Fourier transforms, convolutions, probability mass functions, probability density functions, mean values, variances, vectors, matrices, eigenvalues, and eigenvectors. These topics will *not* be reviewed in the lectures. Instead, they will be probed on Problem Set 1. The supplementary reading for this problem set may help you review, but it is probably better—and easier—if you refer to the course materials you have from wherever and whenever you learned basic signals and systems, probability, and linear algebra.

There is no required text for 6.453. Lecture-by-lecture notes will be provided, along with suggestions for supplementary reading. There will also be some notes distributed, e.g., the probability theory notes being given out today.

There will be eight problem sets, assigned during the first eight weeks of the term. These will be graded and solutions will be distributed. Most, but not all, of these problems will be identical to ones assigned in previous offerings of 6.453. It is expected, however, that any homework you submit represents *your* work. Thus, while it is permissible to discuss the problems with other students in the class, your homework submissions must be your own, and not a team effort or work copied from another student. Likewise, seeking out and making use of the problem set solutions that were distributed in previous years is expressly forbidden.

Grading in 6.453 will be based on the problem sets, the mid-term quiz, and the term papers. Guidelines for term papers will be provided, and you will be required to submit a one paragraph proposal. To ensure that you have decided on a topic far enough in advance to leave time to prepare your term paper, these proposals will be due the day of the mid-term quiz. Furthermore, it should be noted that term papers are *not* expected to represent original research, but instead present an understanding and appreciation for the technical literature on the topic that was selected. Finally, to ensure that your term paper constitutes a broadening educational experience associated with 6.453, it is *not* appropriate to use background reading from your thesis, or some other research project in which you are already engaged, as the topic for your 6.453 term paper.

In addition to the lecture notes, each lecture will be accompanied by a set of slides that will be distributed at the start of that class. The notes and the slides are *not* sufficient to substitute for lecture attendance, but instead are meant to simplify note taking during class. Course materials—lecture notes, slides, problem sets, and problem set solutions—will be available.

Technical Overview

The rest of today's lecture will be devoted to placing 6.453 in its proper context, i.e., by showing where classical and quantum physics part company in the context of optical communications. In particular, without giving you sufficient details to understand how and why these things can be accomplished—after all that's what this entire semester-long class will be about—we will highlight three purely quantum phenomena of relevance to quantum optical communication: quadrature noise squeezing, polarization entanglement, and teleportation.

Quadrature Noise Squeezing

Let's start with the simplistic semiclassical description of optical homodyne detection, shown on Slide 3. Semiclassical photodetection refers to the theory of light detection in which the electromagnetic field is described via classical physics, and the fundamental noise that limits the sensitivity with which weak light fields can be measured is the shot noise associated with the discreteness (quantum nature) of the electron charge. As shown on Slide 3, a weak signal field, represented as a single-frequency signal with complex amplitude a_s and carrier frequency ω is combined—at a lossless 50/50 beam splitter—with a strong single-frequency local-oscillator field with complex amplitude a_{LO}, where $|a_{\text{LO}}|^2 \gg |a_s|^2$, of the same carrier frequency. Here we have ignored the electromagnetic polarization and spatial characteristics of these fields, to keep the notation as simple as possible, although such considerations figure very strongly into physical implementations of optical homodyne detection.

A lossless 50/50 beam splitter can be thought of as a partly-silvered mirror—although the ones in use are usually dielectric mirrors—that transmits half the optical energy incident on either of its input ports and reflects the other half. For our single-frequency waves, we can regard $|a_s|^2$ and $|a_{\text{LO}}|^2$ as the energies arriving at the two input ports. The resulting complex amplitudes at the output ports, a_+ and a_-, can then be taken to satisfy

$$a_\pm = \frac{a_s \pm a_{\text{LO}}}{\sqrt{2}}. \tag{1}$$

It is left as an exercise for you to verify that this input-output relation conserves energy, viz.,

$$|a_+|^2 + |a_-|^2 = |a_s|^2 + |a_{\text{LO}}|^2, \tag{2}$$

for arbitrary values of a_s and a_{LO}, as must be the case because the beam splitter is *passive* as well as lossless.

The fields emerging from the beam splitter's output ports illuminate a pair of photodetectors, resulting in output currents $i_\pm(t)$ that are subsequently combined in a gain-K differential amplifier to obtain

$$K\Delta i(t) \equiv K[i_+(t) - i_-(t)]. \tag{3}$$

This photodetection arrangement is called balanced homodyne detection, where balanced denotes taking the differential output from two photodetectors, and homodyne arises from each photodetector being a low-pass square-law device so that its output current is the baseband beat between the identical-carrier-frequency signal and local oscillator fields.

To make life notationally simple—albeit at odds with what we will see later in the semester—Slide 3 states that the photocurrents $i_\pm(t)$ are statistically independent Poisson random variables with mean values $|a_\pm|^2$, respectively.[1] Recall that a Poisson random variable N with mean m has the probability mass function,

$$P_N(n) = \frac{m^n e^{-m}}{n!}, \quad \text{for } n = 0, 1, 2, \ldots, \tag{4}$$

so that its mean value[2] $\langle N \rangle = m$ coincides with its variance $\text{var}(N) = m$, where

$$\text{var}(N) \equiv \langle (\Delta N)^2 \rangle, \quad \text{with } \Delta N \equiv N - \langle N \rangle. \tag{5}$$

[1]Really, Slide 3 should have said that $q^{-1}\int dt\, i_\pm(t)$, where q is the electron charge, are statistically independent Poisson random variables with those mean values.

[2]We use angle brackets to denote the ensemble average of a classical random variable, which is a convenient choice because the same notation will serve us later for the ensemble-averaged outcome of a quantum measurement.

Thus the signal-to-noise ratio (SNR) in the Poisson random variable—defined to be the square of its mean value divided by its variance (mean-squared fluctuation strength in that random variable) is

$$\mathrm{SNR} = \frac{\langle N\rangle^2}{\langle(\Delta N)^2\rangle} = m = \langle N\rangle. \tag{6}$$

Applying these Poisson random variable results to the semiclassical balanced homodyne detection setup on Slide 3, we get

$$\mathrm{SNR}_{\text{balanced homodyne}} = \frac{4[\mathrm{Re}(a_s a_{\mathrm{LO}}^*)]^2}{|a_{\mathrm{LO}}|^2} \le 4|a_s|^2, \tag{7}$$

with equality if and only if a_s and a_{LO} are in phase, and the derivation being left as another exercise for the reader.

As an application of the preceding balanced homodyne detection scheme, turn now to Slide 4. Here we have a fused fiber coupler with inputs—single-frequency fields with their electromagnetic polarizations, spatial mode characteristics, and common carrier frequencies suppressed—denoted $a_{s_{\mathrm{in}}}$ and $a_{t_{\mathrm{in}}}$, indicating *signal* and *tap* inputs, and similarly denoted outputs $a_{s_{\mathrm{out}}}$ and $a_{t_{\mathrm{out}}}$. In a benign setting, $a_{s_{\mathrm{in}}}$ is a signal-bearing field in a fiber-optic communication network and $a_{t_{\mathrm{in}}} = 0$. The fused-fiber coupler is a lossless asymmetric beam splitter, so that

$$a_{s_{\mathrm{out}}} = \sqrt{T}\, a_{s_{\mathrm{in}}} + \sqrt{1-T}\, a_{t_{\mathrm{in}}} \quad \text{and} \quad a_{t_{\mathrm{out}}} = \sqrt{1-T}\, a_{s_{\mathrm{in}}} - \sqrt{T}\, a_{t_{\mathrm{in}}}, \tag{8}$$

where the transmissivity T satisfies $0 < T < 1$. (The reader should check that this transformation—like its 50/50 special case, $T = 1/2$—conserves energy.) Balanced homodyne detection is performed on the signal and tap output ports, so that two receivers can each view the information that is carried by the input signal $a_{s_{\mathrm{in}}}$. For reference, we note that had balanced homodyne detection been performed on the signal *input* then the resulting signal-to-noise ratio would have been[3]

$$\mathrm{SNR}_{\mathrm{in}} = 4|a_{s_{\mathrm{in}}}|^2. \tag{9}$$

It is easily shown that the SNRs at the signal and tap output ports are

$$\mathrm{SNR}_{\mathrm{out}} = 4T|a_{s_{\mathrm{in}}}|^2 \quad \text{and} \quad \mathrm{SNR}_{\mathrm{tap}} = 4(1-T)|a_{s_{\mathrm{in}}}|^2, \tag{10}$$

respectively. These are intuitively pleasing results. The SNR of balanced homodyne detection equals four times the signal energy that is incident on the detector, and the lossless, passive, fused fiber coupler—with no input applied to its tap-input port—divides the incoming signal energy between the signal output and tap output ports, hence

$$\mathrm{SNR}_{\mathrm{out}} + \mathrm{SNR}_{\mathrm{tap}} = \mathrm{SNR}_{\mathrm{in}}. \tag{11}$$

[3]Here, and in the tap calculations that follow, we will assume that the local-oscillator phase has been chosen to match that of the field undergoing homodyne detection.

The noise in balanced homodyne detection is shot noise associated with the discrete nature of the electron charge and the internal workings of the two photodetectors. So, how could anything other than Eq. (11) occur? The answer, of course, is quantum mechanics!

The left-hand side of Slide 5 shows the simplistic quantum treatment of balanced homodyne detection. Now, the complex amplitudes are replaced by quantum-mechanical operators—denoted by carets—whose associated quantum states determine the measurement statistics that will be obtained. With the signal and local oscillator quantum states coming from laser light—something we will learn about as *coherent states*—the homodyne SNR is still given by $4|a_{s_{\rm in}}|^2$ at the input to the fused fiber coupler. Now however, because quantum photodetection holds that the noise seen in high-sensitivity photodetection is due to the quantum state of the illuminating field, and *not* to any shot effect associated with the discreteness of the electron charge, it is possible to choose an input quantum state for the tap such that *both* the signal and tap outputs achieve homodyne SNRs that approach $4|a_{s_{\rm in}}|^2$ for all $0 < T < 1$. Evidently, with the right quantum "magic," which in this case we will see is quadrature noise squeezing, we can have our SNR cake and eat it too!

Here is a quick once over on how the preceding—and astounding—SNR results are obtained. First, we need to pin the down the inputs used for the signal and tap fields and for the local oscillators in the balanced homodyne detectors. The operator-valued input-output relation for the fused-fiber coupler is

$$\hat{a}_{s_{\rm out}} = \sqrt{T}\,\hat{a}_{s_{\rm in}} + \sqrt{1-T}\,\hat{a}_{t_{\rm in}} \quad \text{and} \quad \hat{a}_{t_{\rm out}} = \sqrt{1-T}\,\hat{a}_{s_{\rm in}} - \sqrt{T}\,\hat{a}_{t_{\rm in}}. \tag{12}$$

A laser light (coherent-state) input to the coupler has

$$\langle \hat{a}_{s_{\rm in}} \rangle = a_{s_{\rm in}} \quad \text{and} \quad \text{var}[\text{Re}(\hat{a}_{s_{\rm in}})] = \text{var}[\text{Im}(\hat{a}_{s_{\rm in}})] = 1/4, \tag{13}$$

where, for convenience, we will take $a_{s_{\rm in}} = |a_{s_{\rm in}}|$ to be positive real. To get the *non-classical* tap performance, we then use a *squeezed-vacuum* state for the tap input, for which $\langle \hat{a}_{t_{\rm in}} \rangle = 0$, and

$$\text{var}[\text{Re}(\hat{a}_{t_{\rm in}})] = s/4 \quad \text{and} \quad \text{var}[\text{Im}(\hat{a}_{t_{\rm in}})] = 1/4s, \tag{14}$$

with $0 < s < 1$, and we use strong coherent-state local oscillators (in the balanced homodyne detectors) with $\langle \hat{a}_{\rm LO} \rangle = a_{\rm LO}$ that is positive real.

The real and imaginary parts of a field operator correspond to the quadrature components of the carrier-frequency ω field. Thus a coherent state has equal fluctuation strengths in its two quadratures, whose variance product, as we will learn, satisfies the Heisenberg uncertainty principle with equality.[4] The noise in the squeezed-vacuum

[4]The vacuum state is the zero-mean coherent state—which is the quantum version of a field's being "zero"—so it too is an equal fluctuation strength minimum uncertainty state.

state, however, has *unequal* fluctuation strengths in its two quadratures, while maintaining a Heisenberg-limited uncertainty product.[5] Using these results, in conjunction with quantum photodetection theory, gives us

$$\mathrm{SNR}_{\mathrm{out}} = \frac{T|a_{s_{\mathrm{in}}}|^2}{T/4 + (1-T)s/4} \approx 4|a_{s_{\mathrm{in}}}|^2, \tag{15}$$

$$\mathrm{SNR}_{\mathrm{tap}} = \frac{(1-T)|a_{s_{\mathrm{in}}}|^2}{(1-T)/4 + Ts/4} \approx 4|a_{s_{\mathrm{in}}}|^2, \tag{16}$$

where the approximations are valid when s is small enough that the second term in each noise denominator can be neglected. Physically, quantum photodetection tells us that the noise seen in the two balanced homodyne detectors is quantum noise from the real-part quadrature of their illuminating fields. When we "squeeze" that quadrature hard enough at the tap input port, we can make *both* the signal and tap output noises in their real-part quadratures be dominated by their signal light contributions. This, in turn, is why their SNRs can *both* approximate the input SNR.

The preceding development was cast in terms of a benign scenario for a fiber-optic communication network. It could also be applied to a less benign situation, in which an unauthorized person—the nefarious eavesdropper "Eve"—uses a $T \ll 1$ tap to surreptitiously listen in on the information being carried by the signal light. Later this term, when we study quadrature noise squeezing in detail, you will see that this attack is less threatening then it might now appear.

Polarization Entanglement

The shot noise theory (semiclassical photodetection) that proved inadequate to explain quadrature noise squeezing—and the low-loss waveguide tap—comes from classical electromagnetism and the discreteness of the electron charge. Perhaps we can break through the limitation of semiclassical photodetection if we start considering photons as classical entities, i.e., they behave in many ways like classical particles, but they also possess some wave-like properties. As we will now show, this semiquantum view also fails to explain fully quantum behavior.

Consider a monochromatic $+z$-going photon of frequency ω. Aside from an absolute phase factor, such a photon is completely characterized by its electromagnetic polarization state. In terms of electromagnetic wave theory, this refers to the locus of points traced out by the tip of the electric field vector, as a function of time, at an arbitrary point in a constant-z plane. Horizontal (H) polarization means that the electric field oscillates back and forth along the x axis. Likewise, vertical (V) polarization means that the electric field oscillates back and forth along the y axis. Right circular (RC) polarization is when the electric field performs clockwise circular motion in a right-handed x-y plane, and left circular (LC) polarization is when that

[5] Hence s is called the squeeze parameter.

circular motion is counterclockwise.[6] In general, of course, this photon may have elliptical polarization.

One convenient explicit representation for the polarization of our single photon is in terms of a complex-valued, unit-length, 2D column vector

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}, \tag{17}$$

whose elements are the projections of $\mathbf{i}$ onto the x and y axes. The conjugate transpose of $\mathbf{i}$, denoted $\mathbf{i}^\dagger = \begin{bmatrix} \alpha_x^* & \alpha_y^* \end{bmatrix}$, gives us the unit-length condition in the form

$$\mathbf{i}^\dagger \mathbf{i} = |\alpha_x|^2 + |\alpha_y|^2 = 1, \tag{18}$$

and the locus of points traced out by the electric field over time is specified by

$$\mathrm{Re}(\mathbf{i}e^{-j\omega t}) = \mathrm{Re}[(\alpha_x \vec{i}_x + \alpha_y \vec{i}_y)e^{-j\omega t}]. \tag{19}$$

So, for example, we see that

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = H \text{ polarization} \quad \text{and} \quad \mathbf{i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = V \text{ polarization}. \tag{20}$$

Similarly, we have that

$$\mathbf{i} = \begin{bmatrix} 1/\sqrt{2} \\ j/\sqrt{2} \end{bmatrix} = \text{RC polarization} \quad \text{and} \quad \mathbf{i} = \begin{bmatrix} 1/\sqrt{2} \\ -j/\sqrt{2} \end{bmatrix} = \text{LC polarization}. \tag{21}$$

Although some polarization-related calculations are easiest done in terms of the complex-valued 2D unit vector $\mathbf{i}$, for others it is preferable to use an equivalent polarization representation in terms of a real-valued 3D unit vector $\mathbf{r}$.[7] The resulting representation is called the Poincaré sphere, and it is shown on Slide 6.

In terms of the Cartesian components, α_x and α_y, of the complex-valued unit vector $\mathbf{i}$, we define the real-valued unit vector $\mathbf{r}$ as follows:

$$\mathbf{r} = \begin{bmatrix} 2\mathrm{Re}(\alpha_x^* \alpha_y) \\ 2\mathrm{Im}(\alpha_x^* \alpha_y) \\ |\alpha_x|^2 - |\alpha_y|^2 \end{bmatrix}. \tag{22}$$

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[6] Here, we have taken the convention that a right-handed x-y plane is one in which $\vec{i}_z = \vec{i}_x \times \vec{i}_y$, with $\vec{i}_k$ being the unit vector in the k direction, goes *into* the x-y plane.

[7] Because $\mathbf{i}$ is a complex-valued 2D vector, it might seem that 4 independent real numbers are required for its specification. However, only 2 independent real numbers are really needed. The unit-length constraint implies that at most 3 real numbers are needed to specify $\mathbf{i}$. Furthermore, because only the relative phase between α_x and α_y is physically significant, insofar as the polarization state is concerned, we can assume the $\alpha_x = |\alpha_x|$ without loss of generality, so that 2 independent real numbers suffice. The same behavior is evident in the vector $\mathbf{r}$. Its construction depends on the relative phase between α_x and α_y, not on their absolute phases. Moreover, with the unit-length constraint on $\mathbf{r}$, we see that it is determined by 2 independent real numbers, just as we saw for $\mathbf{i}$.

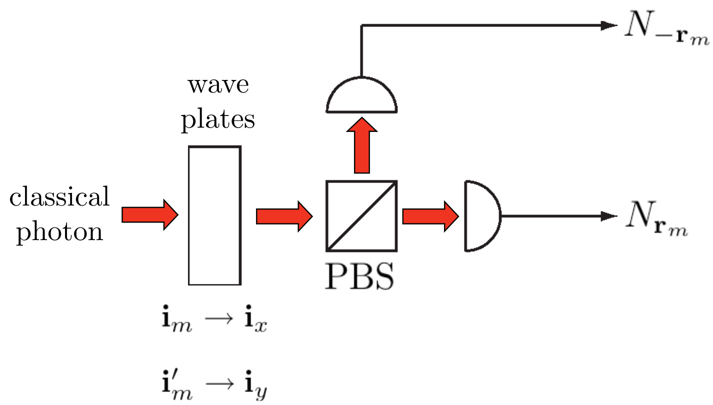

Figure 1: Setup for doing polarization analysis of a classical photon. PBS denotes a polarizing beam splitter, and $N_{\mathbf{r}_m}$ and $N_{-\mathbf{r}_m}$ are the photon counts for the $\mathbf{r}_m \leftrightarrow \mathbf{i}_m$ and $-\mathbf{r}_m \leftrightarrow \mathbf{i}'_m$ polarization basis.

On the Problem Sets 2 and 3 you will explore some of the properties of this polarization representation, including verifying that the polarizations identified on Slide 6 are correct, and showing that the **i** and **r** representations are equivalent. For now, let us inject some wave-particle duality into our treatment of classical photons. We will do this by showing how one does polarization analysis for a single photon. In what follows we will use the **i** and **r** polarization representations interchangeably, choosing whichever one is most convenient for the argument at hand.

An idealized setup for the aforementioned polarization analysis is shown in Fig. 1. Here, we will measure the polarization state of a $+z$-going classical photon in the $\{\mathbf{i}_m, \mathbf{i}'_m\}$ basis, where

$$\mathbf{i}_m \equiv \begin{bmatrix} \alpha_{m_x} \\ \alpha_{m_y} \end{bmatrix} \quad \text{and} \quad \mathbf{i}'_m \equiv \begin{bmatrix} -\alpha^*_{m_y} \\ \alpha^*_{m_x} \end{bmatrix}, \tag{23}$$

for any $|\alpha_{m_x}|^2 + |\alpha_{m_y}|^2 = 1$. Because $\mathbf{i}_m^\dagger \mathbf{i}'_m = 0$, *any* polarization vector can be expressed as a linear combination of $\mathbf{i}_m$ and $\mathbf{i}'_m$.

The wave plates in Fig. 1—see Problem Set 2 for details—are chosen to effect the polarization transformation that turns $\mathbf{i}_m$ and $\mathbf{i}'_m$ into

$$\mathbf{i}_x \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{i}_y \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{24}$$

respectively. The polarizing beam splitter (PBS) transmits horizontal (x) polarization and reflects vertical (y) polarization, so that the single-photon counters, whose outputs are $N_{\mathbf{r}_m}$ and $N_{-\mathbf{r}_m}$, respectively will tell us whether the incoming classical photon was transmitted ($N_{\mathbf{r}_m} = 1$ and $N_{-\mathbf{r}_m} = 0$) or reflected ($N_{\mathbf{r}_m} = 0$ and $N_{-\mathbf{r}_m} = 1$) by the PBS.

Here is where the wave-particle duality comes to the fore. If the $+z$-going classical photon was $\mathbf{i}_m$ polarized, then it will be transformed into $\mathbf{i}_x$ polarization by the wave plates, after which it will be transmitted by the PBS and registered as a count $N_{\mathbf{r}_m} = 1$. If the $+z$-going classical photon was $\mathbf{i}'_m$ polarized, then it will be transformed into $\mathbf{i}_y$ polarization by the wave plates, after which it will be reflected by the PBS and registered as a count $N_{-\mathbf{r}_m} = 1$. What happens, however, if the $+z$-going photon was in a superposition of the $\mathbf{i}_m$ and $\mathbf{i}'_m$ polarizations with non-zero projections along both of those axes? The wave-like nature of the classical photon implies that the wave plates—which are linear devices—will transform those $\mathbf{i}_m$ and $\mathbf{i}'_m$ components of the $+z$-going classical photon into $\mathbf{i}_x$ and $\mathbf{i}_y$ components, respectively. The PBS then transmits the $\mathbf{i}_x$ component and reflects the $\mathbf{i}_y$ component—again wave-like behavior of a linear device—but there is still only *one* classical photon in the system, and hence only one of the single-photon counters can register a count, i.e., either $N_{\mathbf{r}_m} = 1$ and $N_{-\mathbf{r}_m} = 0$ will occur *or* $N_{\mathbf{r}_m} = 0$ and $N_{-\mathbf{r}_m} = 1$ will occur, and the probabilities of these events are

$$\Pr(N_{\pm\mathbf{r}_m} = 1 \text{ and } N_{\mp\mathbf{r}_m} = 0) = \frac{1 \pm \mathbf{r}_m^T\mathbf{r}}{2}, \tag{25}$$

where $\mathbf{r} \leftrightarrow \mathbf{i}$ is the polarization of the $+z$-going classical photon at the input to the polarization analysis setup. Note that the proper interpretation for this probability distribution is as follows. The $+z$-going photon that entered the polarization analysis setup was $\mathbf{i} \leftrightarrow \mathbf{r}$ polarized. The Fig. 1 setup was arranged to measure the $\pm\mathbf{r}_m$ basis, hence its photon-count output tells us whether the incoming photon was counted in the $\mathbf{r}_m$ or the $-\mathbf{r}_m$ polarization. Getting $N_{\mathbf{r}_m} = 1$ is *not* equivalent to saying that the $+z$-going photon entered the Fig. 1 setup in the $\mathbf{r}_m$ polarization. All it says it that photon did *not* enter that Fig. 1 apparatus in the $-\mathbf{r}_m$ polarization.

So far so good, we have wave-particle duality contained within our classical-photon picture. Let's see what happens when two photons are involved. Suppose we have a source that produces a pair of completely correlated, yet randomly polarized, classical photons. In particular this source simultaneously emits a $+z$-going photon (photon 1) with polarization $\mathbf{r}$ and a $-z$-going photon (photon 2) with the orthogonal polarization $-\mathbf{r}$, where $\mathbf{r}$ is equally likely to be anywhere on the Poincaré sphere. What happens if we perform $\pm\mathbf{r}_m$ polarization analysis on both of these photons? For photon 1 we have that

$$\Pr(\text{photon 1 } = \pm\mathbf{r}_m) = \frac{1 \pm \langle \mathbf{r}_m^T\mathbf{r}\rangle}{2}, \tag{26}$$

where $\langle\cdot\rangle$ denotes averaging over the random Poincaré-sphere vector $\mathbf{r}$.[8] If we take $(r = 1, \theta, \phi)$ to be spherical coordinates on the unit sphere, with $\mathbf{r}_m = (1, 0, 0)$, then

[8] We are being a bit cavalier here, and in what follows, with our notation. This equation gives the probabilities that photon 1 leads to an $N_{\pm\mathbf{r}_m} = 1$ and $N_{\mp\mathbf{r}_m} = 0$ event, which, as we noted earlier, does *not* imply that photon 1 entered its polarization analysis apparatus in the $\mathbf{r}_m$ polarization.

$\mathbf{r}$ being completely random is equivalent to saying that the joint probability density of its polar angle θ and its azimuthal angle ϕ is

$$p(\theta, \phi) = \frac{\sin(\theta)}{4\pi}, \quad \text{for } 0 \le \theta \le \pi \text{ and } 0 \le \phi \le 2\pi. \tag{27}$$

It is then a simple exercise to do the averaging necessary to verify that

$$\Pr(\text{photon 1 } = \pm\mathbf{r}_m) = 1/2, \tag{28}$$

regardless of the polarization basis, $\pm\mathbf{r}_m$, being used in the measurement. This is as it should be for a randomly-polarized photon. A similar calculation will prove that

$$\Pr(\text{photon 2 } = \mp\mathbf{r}_m) = 1/2. \tag{29}$$

More interesting behavior occurs when we look at the joint statistics of the polarization measurements made on photons 1 and 2.

We have that

$$\Pr(\text{photon 1 } = \pm\mathbf{r}_m \text{ and photon 2 } = \mp\mathbf{r}_m) = \left\langle \frac{1 \pm \mathbf{r}_m^T\mathbf{r}}{2}\,\frac{1 \mp \mathbf{r}_m^T(-\mathbf{r})}{2} \right\rangle \tag{30}$$

$$= \frac{1 + \langle(\mathbf{r}_m^T\mathbf{r})^2\rangle}{4} = \frac{1}{4} + \frac{1}{4}\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \frac{\cos^2(\theta)\sin(\theta)}{4\pi} = \frac{1}{3}. \tag{31}$$

From this it follows that

$$\Pr(\text{ photon 2 } = \mp\mathbf{r}_m \mid \text{photon 1 } = \pm\mathbf{r}_m\,) = 2/3, \tag{32}$$

showing that the polarization-analysis measurements for photons 1 and 2 are statistically dependent but *not* perfectly correlated. Presumably, this is the strongest dependence that could be expected, because we started out with completely correlated—albeit completely random—polarizations for photons 1 and 2. Ah, but such is not the case, and the reason (of course) is quantum mechanics!

In quantum mechanics we can create a pair of photons whose polarizations are maximally *entangled*.[9] Entanglement is a quantum-mechanical correlation that exceeds the limits of what is possible in classical physics. In particular, we can create a $+z$-going photon and a $-z$-going photon—denoted photons 1 and 2, respectively, as before—such that each photon is randomly polarized, leading to

$$\Pr(\text{photon 1 } = \pm\mathbf{r}_m) = \Pr(\text{photon 2 } = \mp\mathbf{r}_m) = 1/2, \tag{33}$$

[9]Later this semester we will learn how this can be done with spontaneous parametric downconversion.

just as we saw for the classical photons considered above. However, in the quantum case, polarization analyses of two photons whose polarizations are maximally entangled gives[10]

$$\Pr(\text{ photon 2 } = \mp\mathbf{r}_m \mid \text{photon 1 } = \pm\mathbf{r}_m\,) = 1, \tag{34}$$

a result that *cannot* be derived from the classical-photon analysis given earlier. This perfect correlation of photon 1 and 2's polarization analysis measurements occurs even when the measurements are made after the photons have propagated—in opposite directions—over such long distances that neither measurement apparatus could possibly communicate its measurement result—even at light speed—to the other in time to affect that other's measurement result. As a result, if one adopts a classical viewpoint, this perfect measurement correlation would seem to require instantaneous action at a distance, in violation of relativity. Quantum mechanics has no such problem, because the preceding perfect measurement correlation is a consequence of entanglement; no action at a distance is needed for its explanation, as we will see later this semester.

Teleportation

We have one more startling quantum property to introduce in today's lecture: teleportation. Simply stated, for our $+z$-going, frequency-ω, single photon in some arbitrary polarization $\mathbf{i}$, this has to do with how we can safely and completely convey that polarization information from where the photon was created to some distant location. Slide 8 illustrates two fundamental quantum difficulties and one technological difficulty that make this a very challenging problem. Quantum mechanics does not allow the unknown polarization state of a single photon to be perfectly measured. In effect, the best we can do is to use the polarization analysis setup from Fig. 1 to find out—in a probabilistic sense—whether it is more likely that the photon was $\mathbf{i}_m$ or $\mathbf{i}'_m$ polarized. Now, you might argue that we should clone the unknown-polarization photon, i.e., make an enormous number of identical copies, and then use an array of Fig. 1 setups to make repeated measurements for each of a great many different polarization bases and thereby derive an increasingly accurate picture of the original photon's polarization state as the number of different bases and the number of repeated measurements per basis both grow without bound. Unfortunately, this too cannot be done, because quantum mechanics has a no-cloning theorem that forbids our perfectly duplicating the unknown polarization state of a single photon. Another way to get the polarization information from point A to point B would be to let the photon carrying that information simply propagate from A to B. Here there is a technological limitation. The propagation loss in single-mode optical fiber—the best means for long-distance terrestrial photon propagation—is 0.2 dB/km. Thus, if we

[10]Strictly speaking, this equation requires a specific maximally-entangled state, viz., the singlet state, that we shall see later in the semester.

load a single-photon pulse into a 50-km-long fiber, there is only a 10% probability that this photon will emerge at the far end. Yet, there *is* a way to get the unknown polarization state of a single photon from point A to a distant point B: teleportation.

The four steps of the teleportation protocol are shown on Slide 9. In Step 1, Alice (at point A) and Bob (at point B) share some entanglement, i.e., a source that produces a pair of polarization-entangled photons sends one to Alice and the other to Bob. Now, there is a high probability that one or both of these photons won't make it to their destinations, so this process has to be repeated until Alice and Bob can decide—in a subtle way that won't be described here—that they have each received and stored a photon from the same entangled pair. Once this entanglement sharing has occurred, Alice and Bob can proceed to Step 2. Here, Charlie brings Alice a single photon in some polarization state—that is unknown to Alice—and asks her to teleport it to Bob. She completes Step 2 by making an appropriate joint measurement on the photon that Charlie has supplied *and* the one that she received from the entanglement source. The outcome of this measurement is a *classical* result. In particular, it is two classical bits, viz., either 00, 01, 10, or 11. For Step 3 of the protocol, Alice send her measurement bits to Bob via a classical communication channel, which can be regarded as error free. These two bits tell Bob which of four wave-plate transformations he should apply to the photon he had earlier received from the entanglement source. If all the equipment has functioned perfectly, then Bob's transformed photon will be in the same polarization state as the one that Charlie delivered to Alice. There are many remarkable points to be noted here.

- Alice's measurement tells her *nothing* about the polarization state of Charlie's photon: her four possible measurement outcomes are always equally likely to have occurred.

- Alice's measurement destroys the polarization of both her photon and Charlie's, i.e., the teleportation protocol does not violate the no-cloning theorem, even though Bob will end up with a photon whose polarization state matches that of Charlie's, because by then there will not be a photon at Alice's location that contains any information about that polarization state.

- Causality is not violated, because the classical communication channel is light-speed limited.

- Bob learns nothing about the polarization state of Charlie's photon from this protocol, so teleportation does not violate the principle that the unknown polarization state of a single photon cannot be measured.

Another way to sense the incredible nature of entanglement is to suppose that Charlie knows what polarization state he wants Bob to have, i.e., Charlie has a specific but arbitrary $\mathbf{r}$ value in mind. In general, Charlie must send Bob an *infinite* number of classical bits—via a classical channel—for Bob to know the precise value of this real-valued, unit length, 3D vector. Yet, if Alice and Bob have shared an entangled photon pair, and Alice makes the appropriate joint measurement on her photon and Charlie's, she need only send Bob two bits of classical information to enable his transforming his photon into the $\mathbf{r}$ polarization.

Teleportation is an extremely important quantum protocol. If we succeed in building quantum computers, and we want to network them together, then we will need teleportation to communicate the quantum information between that network's nodes.

The Road Ahead

By the end of this semester you will fully understand—and be able to quantitatively analyze—all of the purely quantum phenomena introduced in this lecture. The journey begins in the next two lectures, when we will lay out the fundamentals of Dirac-notation quantum mechanics.