Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 8
Fall 2016

Issued: Thursday, October 27, 2016 **Due:** Thursday, November 3, 2016

Reading: For entanglement and measures of entanglement:

- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Sect. 12.14.
- D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer Verlag, Berlin, 2000), Sects. 3.4 and 3.5.

For qubit teleportation:

- C.C. Gerry and P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005) Sect. 11.3.
- D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer Verlag, Berlin, 2000), Sects. 3.3 and 3.7.

For quadrature teleportation:

- D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer Verlag, Berlin, 2000), Sect. 3.9.

For optimum binary hypothesis testing:

- C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976) Sects. 4.2 and 6.1.

Problem 8.1

Here we will derive the fidelity of a measure-and-prepare approach to qubit transmission. Suppose that Charlie has a single photon whose polarization state is

$$|\psi_C\rangle = \alpha|H\rangle + \beta|V\rangle,$$

where $|\alpha|^2 + |\beta|^2 = 1$ and $|H\rangle$ and $|V\rangle$ denote horizontally-polarized and vertically-polarized single photon states, respectively. Charlie wants to transmit this state to Bob, but Bob is too far away for reliable fiber-optic transmission of that single photon. Instead, Charlie gives his photon to Alice—who is located nearby—for her to measure in the H/V basis using a polarizing beam splitter and unity quantum efficiency photodetectors, as shown in Fig. 1. If Alice gets a click on her H detector, she sends Bob a classical message saying that he should prepare an H photon as his replica of $|\psi_C\rangle$. If Alice gets a click on her V detector, then she sends Bob a classical message saying that he should prepare a V photon as his replica of $|\psi_C\rangle$. Thus, Bob's state after this measure-and-prepare protocol is

$$|\psi_B\rangle = \begin{cases} |H\rangle, & \text{if Alice got an } H \text{ click} \\ |V\rangle, & \text{if Alice got a } V \text{ click.} \end{cases}$$

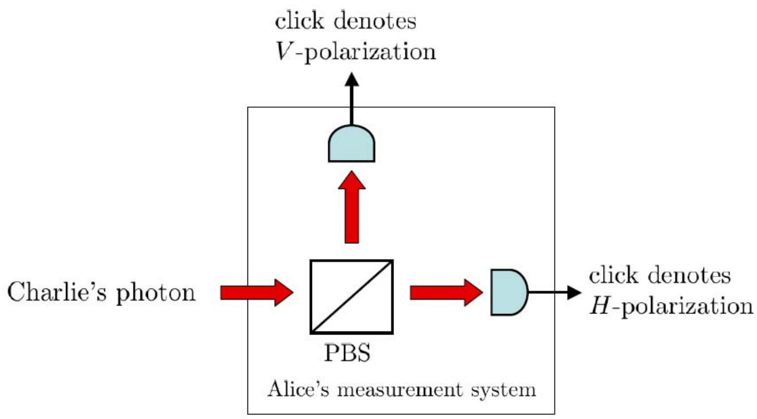

Figure 1: Alice's H/V polarization-measurement system. PBS denotes polarizing beam splitter.

(a) Find $\Pr(\,|\psi_B\rangle = |H\rangle \mid |\psi_C\rangle\,)$ and $\Pr(\,|\psi_B\rangle = |V\rangle \mid |\psi_C\rangle\,)$, i.e., the probabilities for Bob's two possible states conditioned on the value of Charlie's state. Express your answer in terms of α and β.

(b) Use your results from (a) to find $\hat{\rho}_B(\alpha,\beta)$, Bob's density operator when Charlie's state is $|\psi_C\rangle$. Express your answer in terms of α and β.

(c) Use your result from (b) to evaluate the fidelity of the measure-and-prepare system conditioned on the value of Charlie's state, i.e.,

$$F(\alpha,\beta) \equiv \langle\psi_C|\hat{\rho}_B(\alpha,\beta)|\psi_C\rangle.$$

Express your answer in terms of α and β.

(d) Now suppose that Charlie's state is random, and uniformly distributed over the Poincaré sphere, i.e., it has a 3-D unit vector representation,

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \equiv \begin{bmatrix} 2\mathrm{Re}(\alpha^*\beta) \\ 2\mathrm{Im}(\alpha^*\beta) \\ |\alpha|^2 - |\beta|^2 \end{bmatrix},$$

that is uniformly distributed over the unit sphere. Find the numerical value of the measure-and-prepare system's average fidelity, i.e.,

$$\bar{F} \equiv \int_{\mathbf{r}\in\mathcal{P}} \mathrm{d}\mathbf{r}\, \frac{F(\alpha,\beta)}{4\pi}.$$

HINT: Write $F(\alpha,\beta)$ as a function of $\mathbf{r}$, then express $\mathbf{r}$ in spherical coordinates, i.e.,

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}, \quad \text{for } 0 \le \theta \le \pi \text{ and } 0 \le \phi \le 2\pi,$$

and then integrate over the Poincaré sphere using spherical coordinates, viz.,

$$\bar{F} \equiv \int_{\mathbf{r}\in\mathcal{P}} \mathrm{d}\mathbf{r}\, \frac{F(\alpha,\beta)}{4\pi}. = \int_0^\pi \mathrm{d}\theta\, \sin(\theta) \int_0^{2\pi} \mathrm{d}\phi\, \frac{F(\theta,\phi)}{4\pi}.$$

Problem 8.2
Here we will derive the fidelity of a measure-and-prepare approach to continuous-variable quantum communication. Suppose that Charlie has a single-mode field in the state $|\psi_C\rangle$ which he wishes to transmit to Bob. Because Bob is too far away for reliable quantum transmission, Charlie sends his field mode to Alice—who is located nearby—for her to measure via balanced heterodyne detection, i.e., by the positive operator-valued measurement (POVM) associated with the annihilation operator $\hat{a}_C$ of Charlie's field mode. Alice's outcome from this measurement is a complex-valued random variable α. She sends this classical α value to Bob over a classical communication channel and Bob uses this information to prepare a single-mode field in the coherent state $|\alpha\rangle$.

(a) Find $p(\alpha_1,\alpha_2)$, the joint probability density function for α_1 and α_2, the real and imaginary parts of the random variable α, as a function of $|\psi_C\rangle$.

(b) Express the density operator for Bob's state, $\hat{\rho}_B$, in P-representation form as a function of $|\psi_C\rangle$.

(c) Use your result from (b) to find an expression for the fidelity of this measure-and-prepare system, i.e.,

$$F(|\psi_C\rangle) \equiv \langle\psi_C|\hat{\rho}_B|\psi_C\rangle,$$

as an integral involving a function of $|\psi_C\rangle$.

(d) Suppose that Charlie's state is the coherent state $|\alpha_C\rangle$. Use your answer from (c) to obtain a numerical value for the resulting fidelity, $F(|\alpha_C\rangle)$, of the measure-and-prepare system.

An integral of note:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-A(x-b)^2} = \sqrt{\pi/A}, \quad \text{for } A > 0 \text{ and arbitrary } b.$$

Problem 8.3

Here we shall begin a treatment of optimum binary hypothesis testing. Suppose that a quantum system is known to be in either state $|\psi_{-1}\rangle$ or $|\psi_1\rangle$, where $|\psi_{-1}\rangle \neq |\psi_1\rangle$. Let hypothesis H_{-1} denote "state $= |\psi_{-1}\rangle$" and hypothesis H_1 denote "state $= |\psi_1\rangle$." Assume that these two hypotheses are equally likely, i.e., before we make any measurement on the quantum system, it has probability 1/2 of being in state $|\psi_{-1}\rangle$ and probability 1/2 of being in state $|\psi_1\rangle$. Our task is to make a measurement on this system to determine—with the lowest probability of being wrong—whether the system's state was $|\psi_{-1}\rangle$ or $|\psi_1\rangle$ before we make our measurement. (The projection postulate implies that the system's state will likely be changed by our having made a measurement.)

Because we know the system can only be in $|\psi_{-1}\rangle$ or $|\psi_1\rangle$ we can—and we will—limit all our analysis in the reduced Hilbert space,

$$\mathcal{H} \equiv \mathrm{span}(|\psi_{-1}\rangle, |\psi_1\rangle),$$

i.e., to the Hilbert space of kets of the form

$$|\psi\rangle = \alpha|\psi_{-1}\rangle + \beta|\psi_1\rangle,$$

where α and β are complex numbers.

Define a decision operator,

$$\hat{D} \equiv |d_1\rangle\langle d_1| - |d_{-1}\rangle\langle d_{-1}|,$$

where $\{|d_{-1}\rangle, |d_1\rangle\}$ are a pair of *orthonormal* kets on the reduced Hilbert space $\mathcal{H}$. Clearly, $\hat{D}$ is an observable on $\mathcal{H}$. Suppose that we measure $\hat{D}$ on the quantum system under study. If the outcome of this measurement is -1, we will say that the state before the measurement was $|\psi_{-1}\rangle$. If the outcome of this measurement in 1, we will say that the state before the measurement was $|\psi_1\rangle$.

(a) Find the conditional probabilities,

$$\begin{aligned}\Pr(\text{ say "state was } |\psi_{-1}\rangle\text{" } \mid \text{ state was } |\psi_1\rangle\,) &= \Pr(\hat{D} = -1 \mid |\psi_1\rangle),\\ \Pr(\text{ say "state was } |\psi_1\rangle\text{" } \mid \text{ state was } |\psi_{-1}\rangle\,) &= \Pr(\hat{D} = 1 \mid |\psi_{-1}\rangle).\end{aligned}$$

and the unconditional error probability,

$$\begin{aligned}\Pr(e) &\equiv \Pr(\text{state was } |\psi_{-1}\rangle)\Pr(\hat{D}=1 \mid |\psi_{-1}\rangle) \\ &+ \Pr(\text{state was } |\psi_1\rangle)\Pr(\hat{D}=-1 \mid |\psi_1\rangle).\end{aligned}$$

(b) Suppose that $\langle\psi_{-1}|\psi_1\rangle = 0$, so that $\{|\psi_{-1}\rangle, |\psi_1\rangle\}$ is an orthonormal basis for $\mathcal{H}$. Find the measurement eigenkets $\{|d_{-1}\rangle, |d_1\rangle\}$ that minimize your error probability expression from (a). [The error probability of your optimum decision operator for this case shows why orthonormal kets are said to be "distinguishable."]

(c) Suppose that $|\psi_{-1}\rangle$ and $|\psi_1\rangle$ are normalized (unit length), but *not* orthogonal. In particular, let $\{|x\rangle, |y\rangle\}$ be an orthonormal basis for $\mathcal{H}$, and assume that,

$$|\psi_{-1}\rangle = \cos(\theta)|x\rangle - \sin(\theta)|y\rangle \quad \text{and} \quad |\psi_1\rangle = \cos(\theta)|x\rangle + \sin(\theta)|y\rangle,$$

where $0 < \theta < \pi/4$. Using the expansions,

$$|d_{-1}\rangle = \cos(\phi)|x\rangle - \sin(\phi)|y\rangle \quad \text{and} \quad |d_1\rangle = \sin(\phi)|x\rangle + \cos(\phi)|y\rangle,$$

where $0 \le \phi < 2\pi$, and your $\Pr(e)$ result from (a) find the ϕ value—hence the $\{|d_{-1}\rangle, |d_1\rangle\}$—that minimizes the error probability for this case.

[Hint: By assiduous use of trig identities, you should be able to reduce the error probability expression to the following form:

$$\Pr(e) = \frac{1}{2}[1 - \sin(2\phi)\sin(2\theta)],$$

which is easily minimized over ϕ.]

Problem 8.4

Here we shall continue our treatment of optimum binary hypothesis testing for equally-likely hypotheses, H_{-1} = state is $|\psi_{-1}\rangle$ and H_1 = state is $|\psi_1\rangle$. Suppose that the quantum system considered in Problem 8.3 is a single-mode optical field with annihilation operator $\hat{a}$.

(a) Let $|\psi_{-1}\rangle = |n_{-1}\rangle$ and $|\psi_1\rangle = |n_1\rangle$ be photon number states with $n_{-1} \ne n_1$. Show that making the number operator measurement, $\hat{N} \equiv \hat{a}^\dagger\hat{a}$, on the single-mode field allows a zero-error-probability decision to be made as to whether the state before the measurement was $|n_{-1}\rangle$ or $|n_1\rangle$.

(b) Let $|\psi_{-1}\rangle = |\alpha_{-1}\rangle$ and $|\psi_1\rangle = |\alpha_1\rangle$ be coherent states with $\langle\alpha_{-1}|\alpha_1\rangle = \cos(2\theta)$ for a θ value satisfying $0 < \theta < \pi/4$. Find the error probability achieved by the minimum-error-probability decision operator for deciding whether the state before the measurement was $|\alpha_{-1}\rangle$ or $|\alpha_1\rangle$.

(c) Evaluate your error probability from (b) when on-off keying (OOK) is used: $|\alpha_{-1}\rangle = |0\rangle$ and $|\alpha_1\rangle = |\sqrt{N}\rangle$, i.e., when the two coherent states we are trying to distinguish are the vacuum state, and a coherent state with average photon number N. Compare this error probability with what is achieved when we make the $\hat{N}$ measurement and say "state was $|0\rangle$" when this measurement yields outcome 0 and say "state was $|\sqrt{N}\rangle$" when this measurement yields a non-zero outcome.

[<u>Hint</u>: First find the conditional error probabilities,

$$\Pr(\text{ say "state was } |0\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\ ),$$

and

$$\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |\sqrt{0}\rangle\ ).$$

and then find the unconditional error probability using these intermediate results.]

(d) Evaluate your error probability from (b) when binary phase-shift keying (BPSK) is used: $|\alpha_{-1}\rangle = |-\sqrt{N}\rangle$ and $|\alpha_1\rangle = |\sqrt{N}\rangle$. Compare this error probability with what is achieved when we make the $\hat{a}_1 = \text{Re}(\hat{a})$ measurement and say "state was $|-\sqrt{N}\rangle$" when this measurement yields a negative outcome and say "state was $|\sqrt{N}\rangle$" when this measurement yields a non-negative outcome. Express your answer for the homodyne receiver in terms of

$$Q(x) \equiv \int_x^\infty dt\, \frac{e^{-t^2/2}}{\sqrt{2\pi}},$$

i.e., the probability that a zero-mean, unity-variance Gaussian random variable exceeds x.

[<u>Hint</u>: First find the conditional error probabilities,

$$\Pr(\text{ say "state was } |-\sqrt{N}\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\ ),$$

and

$$\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |-\sqrt{N}\rangle\ ).$$

and then find the unconditional error probability using these intermediate results.]

Problem 8.5

Here we shall consider a different variant of the binary hypothesis testing problem. Suppose, as in Problem 8.3, that a quantum system is known to be in either state $|\psi_{-1}\rangle$ or $|\psi_1\rangle$, where $|\psi_{-1}\rangle \neq |\psi_1\rangle$. Let hypothesis H_{-1} denote "state = $|\psi_{-1}\rangle$" and hypothesis H_1 denote "state = $|\psi_1\rangle$." Assume that these two hypotheses are equally likely, i.e., before we make any measurement on the quantum system, it has probability 1/2 of being in state $|\psi_{-1}\rangle$ and probability 1/2 of being in state $|\psi_1\rangle$. Our task is still to make a measurement on this system to determine whether the system's state was $|\psi_{-1}\rangle$ or $|\psi_1\rangle$ before we make our measurement. Now, however, we do not want to make *any* mistakes, i.e., when we say "state was $|\psi_{-1}\rangle$" we must be correct, and when we say "state was $|\psi_1\rangle$" we must also be correct. This does *not* require that we limit ourselves to orthonormal states $|\psi_{-1}\rangle$ and $|\psi_1\rangle$, because we will also allow our measurement outcome to be "error," meaning it cannot reliably determine whether the state was $|\psi_{-1}\rangle$ or $|\psi_1\rangle$. In other words, we will require a measurement on the two-dimensional reduced Hilbert space $\mathcal{H}$ that has three possible outcomes: "state was $|\psi_{-1}\rangle$," "state was $|\psi_1\rangle$," and "error."

Assume that,

$$|\psi_{-1}\rangle = \cos(\theta)|x\rangle - \sin(\theta)|y\rangle \quad \text{and} \quad |\psi_1\rangle = \cos(\theta)|x\rangle + \sin(\theta)|y\rangle,$$

where $0 < \theta < \pi/4$, as in Problem 8.3(c), where $|x\rangle$ and $|y\rangle$ are an orthonormal basis for $\mathcal{H}$. Define a pair of kets,

$$|\xi_{-1}\rangle = -\sin(\theta)|x\rangle + \cos(\theta)|y\rangle \quad \text{and} \quad |\xi_1\rangle = -\sin(\theta)|x\rangle - \cos(\theta)|y\rangle$$

and a set of operators $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$,

$$\begin{aligned}\hat{\Pi}_{-1} &\equiv a|\xi_{-1}\rangle\langle\xi_{-1}|,\\ \hat{\Pi}_1 &\equiv a|\xi_1\rangle\langle\xi_1|,\\ \hat{\Pi}_e &\equiv b|x\rangle\langle x|,\end{aligned}$$

where a and b are positive constants.

(a) Find a and b such that $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$ is a positive operator-valued measure (POVM) on the reduced Hilbert space $\mathcal{H}$, i.e., find the values of a and b for which

$$\hat{\Pi}_j^\dagger = \hat{\Pi}_j, \quad \text{for } j = -1, 1, e,$$

$$\langle\psi|\hat{\Pi}_j|\psi\rangle \geq 0, \quad \text{for } j = -1, 1, e \text{ and all } |\psi\rangle,$$

and

$$\hat{\Pi}_{-1} + \hat{\Pi}_1 + \hat{\Pi}_e = \hat{I}_2,$$

where $\hat{I}_2$ is the identity operator on $\mathcal{H}$.

(b) When we measure $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$—with a and b as found in (a), so that these operators form a POVM and hence represent a measurement—and the state of the quantum system is $|\psi\rangle \in \mathcal{H}$, the outcome will be either -1, 1, or e, with the following probabilities:

$$\begin{aligned}
\Pr(\text{outcome} = -1) &= \langle\psi|\hat{\Pi}_{-1}|\psi\rangle, \\
\Pr(\text{outcome} = 1) &= \langle\psi|\hat{\Pi}_{1}|\psi\rangle, \\
\Pr(\text{outcome} = e) &= \langle\psi|\hat{\Pi}_{e}|\psi\rangle.
\end{aligned}$$

Suppose that we measure this POVM on our quantum system. If the measurement outcome is -1, we will say "state was $|\psi_{-1}\rangle$." If the measurement outcome is 1, we will say "state was $|\psi_1\rangle$." If the measurement outcome is e, we will say "error." Show that this decision procedure will never be incorrect when it says "state was $|\psi_{-1}\rangle$," or when it says "state was $|\psi_1\rangle$."

(c) For the POVM decision rule from (b), find the unconditional error probability, Pr(outcome = "error").

(d) Evaluate your error probability from (c) when $|\psi_{-1}\rangle = |-\sqrt{N}\rangle$ and $|\psi_1\rangle = |\sqrt{N}\rangle$, for $|\pm\sqrt{N}\rangle$ being coherent states.