Wednesday, March 30, 2005
Handout #15
Due: Wednesday, April 6, 2005

Problem Set 6

Problem 6.1 (rational realizations).

(a) Generalize Figure 2 of Chapter 9 to realize any causal rational impulse response $g(D) = n(D)/d(D)$ with $\nu = \max\{\deg n(D), \deg d(D)\}$ memory elements, where $n(D)$ and $d(D)$ are both polynomials in $\mathbb{F}_2[D]$.

(b) By a further generalization, show how to realize a rate-$1/n$ convolutional encoder with causal rational transfer function $\mathbf{g}(D) = \mathbf{n}(D)/d(D)$ with $\nu = \max\{\deg \mathbf{n}(D), \deg d(D)\}$ memory elements, where $\mathbf{n}(D)$ and $d(D)$ are polynomial.

Problem 6.2 (rational = eventually periodic).

Show that a Laurent D-transform $f(D) \in \mathbb{F}_2((D))$ is rational if and only if the corresponding Laurent sequence $\mathbf{f}$ is finite or eventually becomes periodic.

[Hints: (a) show that if a sequence $\mathbf{f}$ is eventually periodic with period P, then its D-transform $f(D)$ can be written as $f(D) = g(D)/(1 - D^P)$, where $g(D)$ is finite; (b) using the results of Problem 6.1(a), show that any causal rational Laurent D-transform $f(D) = n(D)/d(D)$ is the impulse response of a finite-state linear time-invariant system over $\mathbb{F}_2$, and therefore must be finite or eventually periodic.]

Problem 6.3 (input/output properties)

(a) If $\mathbf{y}(D) = u(D)\mathbf{g}(D)$ where $u(D)$ is Laurent and $\mathbf{g}(D) = \{n_j(D)/d_j(D)\}$ is causal and rational, show that $\mathbf{y}(D)$ is an n-tuple of formal Laurent series, $\mathbf{y}(D) \in (\mathbb{F}_2((D)))^n$.

(b) Show that $\mathbf{y}(D)$ is rational if and only if $u(D)$ is rational; *i.e.,* the rational subcode of $\mathcal{C} = \{\mathbf{y}(D) = u(D)\mathbf{g}(D) \mid u(D) \in \mathbb{F}_2((D))\}$ is

$$\mathcal{C}_r = \{\mathbf{y}(D) = u(D)\mathbf{g}(D) \mid u(D) \in \mathbb{F}_2(D)\}.$$

(c) Show that $\mathbf{y}(D)$ is finite if and only if $u(D) = a(D)\mathrm{lcm}\{d_j(D)\}/\gcd\{n_j(D)\}$, where $a(D)$ is finite, $\mathrm{lcm}\{d_j(D)\}$ is the least common multiple of the denominators $d_j(D)$ of the $g_j(D)$, and $\gcd\{n_j(D)\}$ is the greatest common divisor of their numerators.

Problem 6.4 (SPC codes have a 2-state trellis diagram.)

Show that if the (catastrophic) rate-1/1 binary linear convolutional code generated by $g(D) = 1 + D$ is terminated with $\deg u(D) < \mu$, then the resulting code is a $(\mu + 1, \mu, 2)$ SPC code. Conclude that any binary linear SPC code may be represented by a 2-state trellis diagram.

Problem 6.5 (The $(7, 4, 3)$ Hamming code has an 8-state trellis diagram.)

Show that if the (catastrophic) rate-1/1 binary linear convolutional code generated by $g(D) = 1 + D + D^3$ is terminated with $\mu = 4$, then the resulting code is a $(7, 4, 3)$ Hamming code.

Problem 6.6 (Viterbi algorithm decoding of SPC codes)

As shown in Problem 6.4, any $(\mu+1, \mu, 2)$ binary linear SPC block code may be represented by a two-state trellis diagram. Let $\mu = 7$, and let the received sequence from a discrete-time AWGN channel be given by $\mathbf{r} = (0.1, -1.0, -0.7, 0.8, 1.1, 0.3, -0.9, 0.5)$. Perform Viterbi algorithm decoding of this sequence, using the two-state trellis diagram of the $(8, 7, 2)$ SPC code.

Compare and contrast the performance and complexity of VA decoding to that of "Wagner decoding" (Problem 4.6) for this example.