QUIZ 1 SOLUTIONS

Problem Q1.2 (a) Does the binary representation constitute a prefix-free code? Explain. How about the unary and unary-binary codes?

Answer: The binary representation does not constitute a prefix-free code. For example 1 (the representation of 1) is a prefix of 10 (the representation of 2). The unary code is prefix-free since the only proper prefixes of $0^{j-1}1$ are strings of all zeroes.

The unary-binary code is prefix free, as can be seen by the following argument: the code word for j is the concatenation of the unary code for $n = \lfloor \log j \rfloor + 1$ is non-decreasing in j followed by the binary representation for $m = j - 2^{n-1}$. We show that this is not the prefix of the code word for any $j' \neq j$, and represent j' in terms of the corresponding (n', m'). If $n \neq n'$, then the unary codes for n and n' are not prefixes of each other so that the overall code words for j and j' are not prefixes of each other. For $n = n'$, the binary representation of m and m' are distinct and both have the same length $(n-1)$, so again the overall codewords for j and j' are the same length and distinct, so not prefixes of each other.

(b) Show that if an optimum (in the sense of minimum expected length) prefix-free code is chosen for any given pmf (subject to the condition $p_i > p_j$ for $i < j$), the code word lengths satisfy $l_i \leq l_j$ for all $i < j$. Use this to show that for all $j \geq 1$

$$l_j \geq \lfloor \log j \rfloor + 1$$

Answer: First, if $i < j$, then $p_i > p_j$, so $l_i \leq l_j$ for an optimum code (see lemma 2.5.1 in the notes). Thus for any given j, there are at least j code words (including that for j) whose lengths are less than that of j. Now at least one string of length l_j must be unused by codewords for $1, 2, \ldots, j$ (since codewords are required for integers greater than j). Each codeword for $i \leq j$ uses at least one of the strings of length l_j. Thus $(j+1) \leq 2^{l_j}$ so $j < 2^{l_j}$. Taking the log of both sides, $\log j < l_j$. Since l_j is an integer, $\lfloor \log j \rfloor < l_j$, so it follows that

$$\lfloor \log j \rfloor + 1 \leq l_j$$

(c) The asymptotic efficiency of a prefix-free code for the positive integers is defined to be $\lim_{j\to\infty} \frac{\log j}{l_j}$. What is the asymptotic efficiency of the unary-binary code?

Answer: The codeword for j in the unary-binary code uses $n = \lfloor \log j \rfloor + 1$ bits for the unary part and $n - 1$ bits for the binary represenation. Thus $l_j = 2\lfloor \log j \rfloor + 1$. Thus

$$\lim_{j\to\infty} \frac{\log j}{l_j} = \lim_{j\to\infty} \frac{\log j}{2\lfloor \log j \rfloor + 1} = \frac{1}{2}$$

(d) Explain how to construct a prefix-free code for the positive integers where the asymptotic efficiency is 1. Hint: Replace the unary code for the integers $n = \lfloor \log j \rfloor + 1$ in the unary-binary code with a code whose length grows more slowly with increasing n.

Answer: We have already seen that the unary-binary code uses fewer bits for large values of j than the unary code does. Thus, we can replace the unary code for n with a unary-binary code for n. With this change, the length of the codeword for j (with $n = \lfloor \log j \rfloor + 1$) is $2\lfloor \log n \rfloor + 1 + n - 1$. Thus

$$l_j = \lfloor \log j \rfloor + 1 + 2\lfloor \log[2\lfloor \log j \rfloor + 1] \rfloor$$

Since the loglog term is negligible compared to the log term, the efficiency is 1.

Problem Q1.3 (True or False) For each of the following, state whether the statement is true or false and briefly indicate your reasoning. No credit will be given without a reason, but considerable partial credit might be given for an incorrect answer that indicates good understanding.

(a) Suppose X and Y are binary-valued random variables with pmf given by $p_X(0) = 0.2$, $p_X(1) = 0.8$ $p_Y(0) = 0.4$ and $p_Y(1) = 0.6$. The joint PMF that maximizes the joint entropy H(X,Y) is given by

$p_{X,Y}(\cdot,\cdot)$	X=0	X=1
Y=0	0.08	0.32
Y=1	0.12	0.48

Answer: True. It can be seen from the table that X and Y are statistically independent random variables. You saw in the homework (exercise 2.16 in the notes) that $H(XY) \leq H(X)+H(Y)$ with equality if and only if X and Y are independent. Thus the independent joint distribution maximizes the joint entropy.

(b) For a DMS source X with alphabet $\mathcal{X} = \{1, 2, \ldots, M\}$, let $L_{\min,1}$, $L_{\min,2}$, and $L_{\min,3}$ be the normalized average length in bits per source symbol for a Huffman code over $\mathcal{X}$, $\mathcal{X}^2$ and $\mathcal{X}^3$ respectively. Then there exists a specific PMF for source X for which $L_{\min,3} > \frac{2}{3}L_{\min,2} + \frac{1}{3}L_{\min,1}$.

Answer: True. One choice for a code mapping blocks of 3 source symbols into variable length code words is to concatenate a Huffman code for two symbols with a Huffman code for the following single symbol. The expected length of this code (encoding 3 source symbols) is $2L_{\min,2}+L_{\min,1}$. Thus the expected length in bits per source symbol is $\frac{2}{3}L_{\min,2}+\frac{1}{3}L_{\min,1}$.

(c) Assume that a continuous valued rv Z has a probability density that is 0 except over the interval $[-A, +A]$. Then the differential entropy $h(Z)$ is upper bounded by $1+\log_2 A$. Also $h(Z) = 1 + \log_2 A$ if and only if Z is uniformly distributed between $-A$ and $+A$.

Answer: True. This is very similar to exercise 3.4. In particular, let $f_Z(z)$, be an arbitrary density that is non-zero only within $[-A, A]$. Let $\overline{f}(z) = \frac{1}{2A}$ for $-A \leq z \leq A$ be the uniform density within $[-A, A]$. For the uniform density, $h(\overline{Z}) = \log(2A)$. To compare

$h(Z)$ for an arbitrary density with $h(\overline{Z})$, note that $h(Z) - \log 2A$ is given by

$$\begin{aligned} h(Z) - \log 2A &= \int_{-A}^{A} f_Z(z) \log[\overline{f}(z)/f_Z(z)]\, dz \\ &= \log(e) \int_{-A}^{A} f_Z(z) \ln[\overline{f}(z)/f_Z(z)]\, dz \\ &\leq \log(e) \int_{-A}^{A} [\overline{f}(z) - f_Z(z)]\, dz = 0 \end{aligned}$$

There is strict inequality above unless $\overline{f}_Z(z) = f_Z(z)$ everywhere (or almost everywhere if you want to practice your new-found measure theory).