MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Recitation 9

Exercise 1.

(a) If $X_1 \sim Cauchy(0, \gamma_1)$, $X_2 \sim Cauchy(0, \gamma_2)$, and they are independent, then $X_1 + X_2 \sim Cauchy(0, \gamma_1 + \gamma_2)$.

(b) If $X \sim Cauchy(0, \gamma)$, then $\alpha X \sim Cauchy(0, \alpha\gamma)$, for all $\alpha > 0$.

(c) Let $\{X_n\}$ be a sequence of i.i.d. random variables, with $X_1 \sim Cauchy(0, \gamma)$. Then,

$$\frac{X_1 + \cdots + X_n}{n} \sim Cauchy(0, \gamma),$$

for all n.

Solution:

(a) We have

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t) = \exp\left(-\gamma_1|t|\right)\exp\left(-\gamma_2|t|\right) = \exp\left(-(\gamma_1+\gamma_2)|t|\right),$$

which corresponds to a $Cauchy(0, \gamma_1 + \gamma_2)$.

(b) We have

$$\phi_{\alpha X}(t) = \phi_X(\alpha t) = \exp\left(-\alpha\gamma|t|\right),$$

which corresponds to a $Cauchy(0, \alpha\gamma)$.

(c) We have

$$\begin{aligned} \phi_{\frac{X_1+\cdots+X_n}{n}}(t) &= \prod_{k=1}^{n} \phi_{X_k}\left(\frac{t}{n}\right)^n \\ &= \prod_{k=1}^{n} \exp\left(-\gamma\left|\frac{t}{n}\right|\right) \end{aligned}$$

which corresponds to a $Cauchy(0, \gamma)$ for all n.

Exercise 2. Let $\{X_n\}$ be a sequence of random variables, such that $\mathbb{E}[X_n] = 0$ and $Var(X_n) \leq \sigma^2$ for all n, and such that $Cov(X_i, X_j) \to 0$ when $|i - j| \to \infty$. Then,

$$S_n = \frac{X_1 + \cdots + X_n}{n} \xrightarrow{i.p.} 0.$$

Solution: For any $\epsilon > 0$, Chebyshev's inequality implies that

$$\mathbb{P}\left(|S_n| \geq \epsilon\right) \leq \frac{Var(S_n)}{\epsilon^2} = \frac{1}{n^2\epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j).$$

Since $Cov(X_i, X_j) \to 0$ when $|i - j| \to \infty$, then for every $\delta > 0$, there exists N_δ such that $|Cov(X_i, X_j)| \leq \delta$ for all i, j such that $|i - j| > N_\delta$. Thus, we have

$$\begin{aligned}
\frac{1}{n^2\epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j) &= \frac{1}{n^2\epsilon^2} \sum_{i=1}^{n} \left(\sum_{j:|i-j|\leq N_\delta} Cov(X_i, X_j) + \sum_{j:|i-j|>N_\delta} Cov(X_i, X_j) \right) \\
&\leq \frac{1}{n^2\epsilon^2} \sum_{i=1}^{n} \left(\sum_{j:|i-j|\leq N_\delta} \sigma^2 + \sum_{j:|i-j|>N_\delta} \delta \right) \\
&\leq \frac{1}{n^2\epsilon^2} \left[n(2N_\delta + 1)\sigma^2 + n^2\delta \right] \\
&\leq \frac{2N_\delta + 1}{n\epsilon^2} + \frac{\delta}{\epsilon^2}.
\end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(|S_n| \geq \epsilon\right) \leq \frac{\delta}{\epsilon^2}.$$

Finally, since this is true for all $\delta > 0$, we get

$$\lim_{n\to\infty} \mathbb{P}\left(|S_n| \geq \epsilon\right) = 0.$$

Exercise 3. Let $\{X_n\}$ be a sequence of i.i.d. random variables such that $X_1 \sim \mathcal{N}(0,1)$. Let us define $Y_k = X_1 + \cdots + X_k$. Show that

$$\frac{Y_1 + \cdots + Y_n}{n^{3/2}} \xrightarrow{d} \mathcal{N}(0, 1/3).$$

Solution: Let us define

$$S_n = \sum_{k=1}^{n} Y_k.$$

Note that

$$S_n = \sum_{k=1}^{n} (n-k+1) X_k.$$

Then, we have

$$\begin{aligned}
\phi_{S_n}(t) &= \prod_{k=1}^{n} \phi_{X_k}(tk) \\
&= \prod_{k=1}^{n} \exp\left(-\frac{(tk)^2}{2}\right) \\
&= \exp\left(-\frac{t^2}{2}\sum_{k=1}^{n} k^2\right) \\
&= \exp\left[-\frac{t^2}{2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right],
\end{aligned}$$

and thus

$$\begin{aligned}
\phi_{\frac{S_n}{n^{3/2}}}(t) &= \phi_{S_n}\left(\frac{t}{n^{3/2}}\right) \\
&= \exp\left[-\frac{t^2}{2}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right].
\end{aligned}$$

Finally, if $S = \lim_{n\to\infty} \frac{S_n}{n^{3/2}}$, we have

$$\begin{aligned}
\phi_S(t) &= \lim_{n\to\infty} \phi_{\frac{S_n}{n^{3/2}}}(t) \\
&= \exp\left(-\frac{t^2}{6}\right),
\end{aligned}$$

which corresponds to a $\mathcal{N}(0, 1/3)$.