MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Recitation 11

Exercise 1. Consider a discrete-time, finite-state Markov chain $\{X_t\}$, with states $\{1, \ldots, n\}$, and transition probabilities p_{ij}. States 1 and n are absorbing, that is, $p_{11} = 1$ and $p_{nn} = 1$. All other states are transient. Let A_1 be the event that the state eventually becomes 1. For any possible starting state i, let $a_i = \mathbf{P}(A_1 \mid X_0 = i)$ and assume that $a_i > 0$ for every $i \neq n$. Conditional on the information that event A_1 occurs, is the process X_n necessarily Markov? If yes, provide a proof, together with a formula for its transition probabilities. If not, provide a counterexample.

Solution: The answer is yes. Let B be an event of the form

$$B = \{X_0 = i_0, X_1 = i_1, \ldots, X_{t-1} = i_{t-1}\}.$$

It suffices to show that the transition probability $\mathbb{P}(X_{t+1} = j \mid X_t = i, A_1, B)$ is unaffected by the past history (the event B). We have

$$\mathbb{P}(X_{t+1} = j \mid X_t = i, A_1, B) = \frac{\mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i, B)}{\mathbb{P}(A_1 \mid X_t = i, B)}.$$

By the Markov property of the process $\{X_t\}$ (the future is independent of the past, given the present), we have

$$\mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i, B) = \mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i),$$

and

$$\mathbb{P}(A_1 \mid X_t = i, B) = \mathbb{P}(A_1 \mid X_t = i),$$

from which the desired result follows.

Furthermore,

$$\begin{aligned}
\mathbb{P}(X_{t+1} = j \mid X_t = i, A_1) &= \frac{\mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i)}{\mathbb{P}(A_1 \mid X_t = i)} \\
&= \frac{\mathbb{P}(A_1 \mid X_t = i, X_{t+1} = j)\mathbb{P}(X_{t+1} = j \mid X_t = i)}{\mathbb{P}(A_1 \mid X_t = i)} \\
&= \frac{p_{ij} a_j}{a_i}.
\end{aligned}$$