PRODUCT MEASURE AND FUBINI'S THEOREM

Contents

In elementary math and calculus, we often interchange the order of summation and integration. The discussion here is concerned with conditions under which this is legitimate.

1 PRODUCT MEASURE

Consider two probabilistic experiments with probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, respectively. We are interested in forming a probabilistic model of a "joint experiment" in which the original two experiments are carried out independently.

1.1 The sample space of the joint experiment

If the first experiment has an outcome ω_1, and the second has an outcome ω_2, then the outcome of the joint experiment is the pair (ω_1, ω_2). This leads us to define a new sample space $\Omega = \Omega_1 \times \Omega_2$.

1.2 The σ-algebra of the joint experiment

Next, we need a σ-algebra on Ω. If $A_1 \in \mathcal{F}_1$, we certainly want to be able to talk about the event $\{\omega_1 \in A_1\}$ and its probability. In terms of the joint experiment, this would be the same as the event

$$A_1 \times \Omega_1 = \{(\omega_1, \omega_2) \mid \omega_1 \in A_1,\ \omega_2 \in \Omega_2\}.$$

Thus, we would like our σ-algebra on Ω to include all sets of the form $A_1 \times \Omega_2$, (with $A_1 \in \mathcal{F}_1$) and by symmetry, all sets of the form $\Omega_1 \times A_2$ (with $(A_2 \in \mathcal{F}_2)$. This leads us to the following definition.

Definition 1. *We define $\mathcal{F}_1 \times \mathcal{F}_2$ as the smallest σ-algebra of subsets of $\Omega_1 \times \Omega_2$ that contains all sets of the form $A_1 \times \Omega_2$ and $\Omega_1 \times A_2$, where $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$.*

Note that the notation $\mathcal{F}_1 \times \mathcal{F}_2$ is misleading: this is not the Cartesian product of $\mathcal{F}_1$ and $\mathcal{F}_2$!

Since σ-fields are closed under intersection, we observe that if $A_i \in \mathcal{F}_i$, then $A_1 \times A_2 = (A_1 \times \Omega_2) \cap (\Omega_1 \cap A_2) \in \mathcal{F}_1 \times \mathcal{F}_2$. It turns out (and is not hard to show) that $\mathcal{F}_1 \times \mathcal{F}_2$ can also be defined as the smallest σ-algebra containing all sets of the form $A_1 \times A_2$, where $A_i \in \mathcal{F}_i$. Alternatively, suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are generated by algebras $\mathcal{F}_{0,1}, \mathcal{F}_{0,2}$. That is $\mathcal{F}_i = \sigma(\mathcal{F}_{0,i}), i = 1, 2$. Then $\mathcal{F}_1 \times \mathcal{F}_2$ is also the smallest σ-algebra containing all sets of the form $A_1 \times A_2$, where $A_i \in \mathcal{F}_{0,i}$.

In the sequel, we will talk about $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$ – measurable functions with respect to $\mathcal{F}_1 \times \mathcal{F}_2$. Recall, this means that for any Borel set $B \subset \mathbb{R}$, the set $\{(\omega_1, \omega_2) \mid g(\omega_1, \omega_2) \in B\}$ belongs to the σ-algebra $\mathcal{F}_1 \times \mathcal{F}_2$. As a practical matter, it is enough to verify that for any scalar c, the set $\{(\omega_1, \omega_2) \mid g(\omega_1, \omega_2) \leq c\}$ is measurable. Other than using this definition directly, how else can we verify that such a function g is measurable? The basic tools at hand are the following:

(a) continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ are measurable;

(b) indicator functions of measurable sets are measurable;

(c) combining measurable functions in the usual ways (e.g., adding them, multiplying them, taking limits, etc.) results in measurable functions.

The following proposition gives further information about $\mathcal{F}_1 \times \mathcal{F}_2$ and functions measurable with respect to it.

Proposition 1. *Let $E \in \mathcal{F}_1 \times \mathcal{F}_2$ then for every $\omega_1 \in \Omega_1$ the set*

$$E_{\omega_1} \triangleq \{\omega_2 \mid (\omega_1, \omega_2) \in E\}$$

belongs to $\mathcal{F}_2$. Consequently, for every $\mathcal{F}_1 \times \mathcal{F}_2$-measurable function f and every ω_1 the function

$$f_{\omega_1}(\omega_2) \triangleq f(\omega_1, \omega_2)$$

is $\mathcal{F}_2$-measurable.

Remark: E_{ω_1} and f_{ω_1} are called slicec of E and f at ω_1, respectively.

Proof. Fix some ω_1 and define a collection of sets

$$\mathcal{L} = \{E \in \mathcal{F}_1 \times \mathcal{F}_2 \mid E_{\omega_1} \in \mathcal{F}_2\}.$$

When $E = A_1 \times A_2$ the set E_{ω_1} is either empty or equal to A_2. Thus $\mathcal{L}$ contains all the rectangles. On the other hand, for any sequence E_j we have

$$(\cup_j E_j)_{\omega_1} = \bigcup_j (E_j)_{\omega_1}$$

and

$$(E^c)_{\omega_1} = (E_{\omega_1})^c.$$

Thus $\mathcal{L}$ is closed under countable unions and complements. Hence $\mathcal{L}$ is a σ-algebra, which by minimality of $\mathcal{F}_1 \times \mathcal{F}_2$ must be equal to the latter. This shows these statement for sets.

Next, a slice of a simple function

$$f = \sum_{i=1}^{N} a_i 1_{E_i}$$

at ω_1 is itself a simple (hence measurable) function on $(\Omega_2, \mathcal{F}_2)$. This follows from what was just shown for slices of sets. For the general f we have $f = \lim_{r\to\infty} f_r$, where f_r are simple functions. Since the slice of each f_r is $\mathcal{F}_2$ measurable and the class of $\mathcal{F}_2$-measurable functions is closed under taking limits the result follows. $\square$

1.3 The product measure

We now define a measure, to be denoted by $\mathbb{P}_1 \times \mathbb{P}_2$ (or just $\mathbb{P}$, for short) on the measurable space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$. To capture the notion of independence, we require that

$$\mathbb{P}(A_1 \times A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2), \qquad \forall\, A_1 \in \mathcal{F}_1,\ A_2 \in \mathcal{F}_2. \tag{1}$$

Theorem 1. *There exists a unique measure $\mathbb{P}$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ that has property (1). Furthermore, for every $E \in \mathcal{F}_1 \times F_2$ measure $\mathbb{P}(E)$ satisfies*

$$\mathbb{P}(E) = \int \mathbb{P}_2(E_{\omega_1})\,\mathbb{P}_1(d\omega_1) \tag{2}$$

$$= \int \mathbb{P}_1(E_{\omega_2})\,\mathbb{P}_2(d\omega_2). \tag{3}$$

Proof. Uniqueness follows from the fact that $A_1 \times A_2$ is a generating p-system for $\mathcal{F}_1 \times \mathcal{F}_2$ (see Proposition 1 in Lecture 2). We only need to show existence. We start by showing that for every $E \in \mathcal{F}_1 \times \mathcal{F}_2$ the function

$$f_E(\omega_1) \triangleq \mathbb{P}_2(E_{\omega_1})$$

is $\mathcal{F}_1$-measurable. Note that $\mathbb{P}_2(E_{\omega_1})$ is well-defined by Proposition 1. Define a collection

$$\mathcal{L} = \{E : f_E \text{ is } \mathcal{F}_1\text{-measurable}\}.$$

When $E = A_1 \times A_2$ the function $f_E(\omega_1) = \mathbb{P}_2(A_2)1_{A_1}(\omega_1)$, which is clearly measurable. Thus $\mathcal{L}$ contains all rectangles. Next, if E and F are disjoint then so are E_{ω_1} and F_{ω_1}. Consequently,

$$f_{E\cup F}(\omega_1) = f_E(\omega_1) + f_F(\omega_2) \qquad \text{if } E \cap F = \emptyset\,. \tag{4}$$

This implies that $\mathcal{L}$ contains all finite unions of disjoint rectangles. The latter is an algebra of sets (since $(A_1 \times A_2)^c$ can be written as disjoint union of 3 rectangles). Finally, if $E_j \nearrow E$ and $E_j \in \mathcal{L}$ then

$$f_{E_j} \nearrow f_E \tag{5}$$

and therefore f_E is $\mathcal{F}_1$-measurable. Same argument applies to $E_j \searrow E$. All in all $\mathcal{L}$ is a monotone class, containing an algebra that generates $\mathcal{F}_1 \times \mathcal{F}_2$. So $\mathcal{L} = \mathcal{F}_1 \times \mathcal{F}_2$.

We now *define* for any $E \in \mathcal{F}_1 \times \mathcal{F}_2$

$$\mathbb{P}(E) \triangleq \int f_E(\omega_1)\mathbb{P}_1(d\omega_1)\,. \tag{6}$$

It is evident that this assignment satisfies (1). Finite additivity of $\mathbb{P}$ follows from (4). It remains to show σ-additivity, which in turn is equivalent to continuity. The latter follows from (5) and the MCT.

Thus, existence of $\mathbb{P}$ is established. Furthermore, definition (6) is just a restatement of (2). Regarding (3), construct another measure $\mathbb{P}'$ by exchanging roles of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ in (6). So constructed $\mathbb{P}'$ automatically satisfies (3). Moreover, $\mathbb{P}'$ also verifies (1) and hence coincides with $\mathbb{P}$ on a p-system of rectangles $A \times B$. By Proposition 1 of Lecture 2 we have: $\mathbb{P}' = \mathbb{P}$. $\square$

The above discussion extends to the case of any finite number of probability spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i = 1, 2, \ldots, k$. In particular there exists a unique measure $\mathbb{P}$ on $\Omega = \Omega_1 \times \cdots \times \quad {}_k$ such that for every collection of sets $A_i \in \mathcal{F}_i$,

$$\mathbb{P}(A_1 \times \cdots \times A_k) = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k).$$

The corresponding σ-algebra on Ω is the smallest σ-algebra containing all sets of the form $A_1 \times \cdots \times A_k$ where $A_i \in \mathcal{F}_i$. Moreover, this extends to a countable collections of probability spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i = 1, 2, \ldots$, but now the measure is only defined when a finite collection of the $\{A_i\}$ are not Ω_k, i.e. $i = 1, 2, \ldots, k$

$$\mathbb{P}(A_1 \times \cdots \times A_k \times \quad {}_{k+1} \times \quad {}_{k+2} \times \cdots) = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k).$$

1.4 Beyond probability measures

Everything in these notes extends to the case where instead of probability measures $\mathbb{P}_i$, we are dealing with general measures μ_i, under the assumptions that the measures μ_i are σ-**finite**. (A measure μ is called σ-finite if the set Ω can be partitioned into a countable union of sets, each of which has finite measure.)

The most relevant example of a σ-finite measure is the Lebesgue measure on the real line. Indeed, the real line can be broken into a countable sequence of intervals $(n, n+1]$, each of which has finite Lebesgue measure.

1.5 The product measure on $\mathbb{R}^2$

The two-dimensional plane $\mathbb{R}^2$ is the Cartesian product of $\mathbb{R}$ with itself. We endow each copy of $\mathbb{R}$ with the Borel σ-field $\mathcal{B}$ and one-dimensional Lebesgue measure. The resulting σ-field $\mathcal{B} \times \mathcal{B}$ is called the Borel σ-field on $\mathbb{R}^2$. The resulting product measure on $\mathbb{R}^2$ is called two-dimensional Lebesgue measure, to be denoted here by λ_2. The measure λ_2 corresponds to the natural notion of area. For example,

$$\lambda_2([a, b] \times [c, d]) = \lambda([a, b]) \cdot \lambda([c, d]) = (b - a) \cdot (d - c).$$

More generally, for any "nice" set of the form encountered in calculus, e.g., sets of the form $A = \{(x, y) \mid f(x, y) \leq c\}$, where f is a continuous function, $\lambda_2(A)$ coincides with the usual notion of the area of A.

Remark for those of you who know a little bit of topology – otherwise ignore it. We could define the Borel σ-field on $\mathbb{R}^2$ as the σ-field generated by the collection of open subsets of $\mathbb{R}^2$. (This is the standard way of defining Borel sets in topological spaces.) It turns out that this definition results in the same σ-field as the method of Section 1.2.

2 FUBINI'S THEOREM

Fubini's theorem is a powerful tool that provides conditions for interchanging the order of integration in a double integral. Given that sums are essentially special cases of integrals (with respect to discrete measures), it also gives conditions for interchanging the order of summations, or the order of a summation and an integration. In this respect, it subsumes results such as Corollary 1 at the end of the notes for Lecture 12.

Fubini's theorem holds under two different sets of conditions: (a) nonnegative functions g (compare with the MCT); (b) functions g whose absolute value has a finite integral (compare with the DCT). We state the two versions separately, because of some subtle differences.

The two statements below are taken verbatim from the text by Adams & Guillemin, with minor changes to conform to our notation.

Theorem 2. *Let $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative measurable function. Let $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ be a product measure. Then,*

(a) $\int_{\Omega_2} g(\omega_1, \omega_2)\, d\mathbb{P}_2$ *is a measurable function of* ω_1.

(b) $\int_{\Omega_1} g(\omega_1, \omega_2)\, d\mathbb{P}_1$ *is a measurable function of* ω_2.

(c) We have

$$\int_{\Omega_1} \Big[\int_{\Omega_2} g(\omega_1, \omega_2)\, d\mathbb{P}_2 \Big]\, d\mathbb{P}_1 = \int_{\Omega_2} \Big[\int_{\Omega_1} g(\omega_1, \omega_2)\, d\mathbb{P}_1 \Big]\, d\mathbb{P}_2$$
$$= \int_{\Omega_1 \times \Omega_2} g(\omega_1, \omega_2)\, d\mathbb{P}.$$

Note that some of the integrals above may be infinite, but this is not a problem; since everything is nonnegative, expressions of the form $\infty - \infty$ do not arise.

Proof. For simple functions $g = \sum_{i=1}^n a_i 1_{E_i}, E_i \in \mathcal{F}_1 \times \mathcal{F}_2$ statement

(a) follows from measurability of $\omega_1 \mapsto \mathbb{P}_2(E_{\omega_1})$ established in the proof of Theorem 1. For a general g consider a sequence of simple functions

$$g_r(\omega_1, \omega_2) \nearrow g(\omega_1, \omega_2) \qquad \forall \omega_1, \omega_2$$

as $r \to \infty$. Then we have shown that

$$f_r(\omega_1) = \int_{\ \ 2} g_r(\omega_1, \omega_2)\, d\mathbb{P}_2$$

are $\mathcal{F}_1$ measurable and monotonically increasing $f_r \nearrow f$. By the MCT

$$f(\omega_1) \triangleq \lim_{r\to\infty} \int_{\ \ 2} g_r(\omega_1, \omega_2)\, d\mathbb{P}_2 \tag{7}$$

$$= \int_{\ \ 2} \lim_{r\to\infty} g_r(\omega_1, \omega_2)\, d\mathbb{P}_2 \tag{8}$$

$$= \int_{\ \ 2} g(\omega_1, \omega_2)\, d\mathbb{P}_2. \tag{9}$$

Since f is a limit of measurable f_r's – f must be measurable. By (9) the integral over $\ _2$ is also $\mathcal{F}_1$ measurable. This establishes (a) and (b) by symmetry. Finally (c), for a simple function g is just (2)-(3), while for a general function g we just need to integrate (7) interchanging $\int$ and $\lim$ by the MCT at will. □

Recall now that a function is said to be **integrable** if it is measurable and the integral of its absolute value is finite.

Theorem 3. *Let $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function such that*

$$\int_{\Omega_1 \times \Omega_2} |g(\omega_1, \omega_2)| \, d\mathbb{P} < \infty, \tag{10}$$

where $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$.

(a) For almost all $\omega_1 \in \Omega_1$, $g(\omega_1, \omega_2)$ is an integrable function of ω_2.

(b) For almost all $\omega_2 \in \Omega_2$, $g(\omega_1, \omega_2)$ is an integrable function of ω_1.

(c) There exists an integrable function $h : \Omega_1 \to \mathbb{R}$ such that $\int_{\Omega_2} g(\omega_1, \omega_2) \, d\mathbb{P}_2 = h(\omega_1)$, a.s. (i.e., except for a set of ω_1 of zero $\mathbb{P}_1$-measure for which $\int_{\Omega_2} g(\omega_1, \omega_2) \, d\mathbb{P}_2$ is undefined or infinite).

(d) There exists an integrable function $h : \Omega_2 \to \mathbb{R}$ such that $\int_{\Omega_1} g(\omega_1, \omega_2) \, d\mathbb{P}_1 = h(\omega_2)$, a.s. (i.e., except for a set of ω_2 of zero $\mathbb{P}_2$-measure for which $\int_{\Omega_1} g(\omega_1, \omega_2) \, d\mathbb{P}_1$ is undefined or infinite).

(e) We have

$$\begin{aligned} \int_{\Omega_1} \left[\int_{\Omega_2} g(\omega_1, \omega_2) \, d\mathbb{P}_2 \right] d\mathbb{P}_1 &= \int_{\Omega_2} \left[\int_{\Omega_1} g(\omega_1, \omega_2) \, d\mathbb{P}_1 \right] d\mathbb{P}_2 \\ &= \int_{\Omega_1 \times \Omega_2} g(\omega_1, \omega_2) \, d\mathbb{P}. \end{aligned}$$

Remarks:

1. Both Theorems remain valid when dealing with σ-finite measures, such as the Lebesgue measure on $\mathbb{R}^2$. This provides us with conditions for the familiar calculus formula
$$\int \int g(x, y) \, dx \, dy = \int \int g(x, y) \, dy \, dx.$$

2. In order to apply Theorem 3, we need a practical method for checking the integrability condition (10). Here, Theorem 2 comes to the rescue. Indeed, by Theorem 2, we have
$$\int_{\Omega_1 \times \Omega_2} |g(\omega_1, \omega_2)| \, d\mathbb{P} = \int_{\Omega_1} \int_{\Omega_2} |g(\omega_1, \omega_2)| \, d\mathbb{P}_2 \, d\mathbb{P}_1,$$
so all we need is to work with the right hand side, and integrate one variable at a time, possibly also using some bounds on the way.

Proof. By now converting from a non-negative case to integrable case should be familiar. Theorem 3 is no exception: Given a function g, decompose it into its positive and negative parts, apply Theorem 2 to each part, and in the process make sure that you do not encounter expressions of the form $\infty - \infty$. We omit the details. $\square$

3 SOME CAUTIONARY EXAMPLES

We give a few examples where Fubini's theorem does not apply.

3.1 Nonnegativity and integrability

Suppose that both of our sample spaces are the nonnegative integers: $\Omega_1 = \Omega_2 = \{1, 2, \ldots\}$. The σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ consist of all subsets of Ω_1 and Ω_2, respectively. Then, $\sigma(F_1 \times F_2)$ is composed of all subsets of $\{1, 2, \ldots\}^2$. Let both $\mathbb{P}_1$ and $\mathbb{P}_2$ be the counting measure, i.e. $\mathbb{P}(A) = |A|$. This means that

$$\int_A g \, d\mathbb{P}_1 = \sum_{a \in A} f(a), \qquad \int_B h \, d\mathbb{P}_2 = \sum_{b \in B} h(b),$$

and

$$\int_C f \, d(\mathbb{P}_1 \times \mathbb{P}_2) = \sum_{c \in C} f(c).$$

Consider the function f defined by $f(m, m) = 1$, $f(m, m+1) = -1$, and $f = 0$ elsewhere. It is easier to visualize f with a picture:

$$\begin{array}{ccccc} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

So,

$$\begin{aligned} \int_{\Omega_1} \int_{\Omega_2} f \, d\mathbb{P}_2 \, d\mathbb{P}_1 &= \sum_n \sum_m f(n, m) = 0 \\ &\neq 1 = \sum_m \sum_n f(n, m) = \int_{\Omega_2} \int_{\Omega_1} f \, d\mathbb{P}_1 \, d\mathbb{P}_2. \end{aligned}$$

In this example, the conditions of Fubini's theorem fail to hold: the function f is neither nonnegative nor integrable.

3.2 σ-finiteness

Let $\Omega_1 = (0,1)$, let $\mathcal{F}_1$ be the Borel sets, and let $\mathbb{P}_1$ be the Lebesgue measure. Let $\Omega_2 = (0,1)$ let $\mathcal{F}_2$ be the set of all subsets of $(0,1)$, and let $\mathbb{P}_2$ be the counting measure. In particular, for every infinite (countable or uncountable) subset of $(0,1)$, $\mathbb{P}_2(A) = \infty$.

Let $f(x,y) = 1$ if $x = y$, and $f(x,y) = 0$ otherwise. Then,

$$\int_{\Omega_1} \int_{\Omega_2} f(x,y)\, d\mathbb{P}_2(y)\, d\mathbb{P}_1(x) = \int_{\Omega_1} 1\; d\mathbb{P}_1(y) = 1,$$

but

$$\int_{\Omega_2} \int_{\Omega_1} f(x,y)\, d\mathbb{P}_1(x)\, d\mathbb{P}_2(y) = \int_{\Omega_2} 0\, d\mathbb{P}_2(y) = 0.$$

In this example, the conditions of Fubini's theorem fail to hold: the measure on $(0,1)$ is not σ-finite.