MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018
Problem Set 7

Readings:
Notes from Lectures 11-13.
[GS], Section 4.1-4.8 **and** 5.1-5.2
[Cinlar], Chapter IV.

Exercise 1. (Continuous-discrete Bayes rule) Let K be the number of heads obtained in six (conditionally) independent coins of a biased coin whose probability of heads is itself a random variable Z, uniformly distributed over $[0, 1]$. Find the conditional PDF of Z given K, and calculate $\mathbb{E}[Z \mid K = 2]$. You can use the following formula,

$$\int_0^1 y^\alpha (1-y)^\beta \, dy = \frac{\alpha! \, \beta!}{(\alpha+\beta+1)!},$$

known to be valid for positive integer α and β.

Solution: We have $\mathbb{P}(K = 2 \mid Z = z) = cz^2(1-z)^4$, where c is a normalizing constant. Using Bayes' rule, we have

$$f_Z(z|K=2) = \frac{P(K=2|Z=z)f_Z(z)}{P(K=2)} = \frac{z^2(1-z)^4}{\int_0^1 t^2(1-t)^4 \, dt} 1_{z\in[0,1]}$$

Thus,

$$\mathbb{E}[Z \mid K = 2] = \frac{\int_0^1 z^3(1-z)^4 \, dz}{\int_0^1 t^2(1-t)^4 \, dt} = \frac{3}{8}.$$

Exercise 2. Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X+Y$ and $V = X/(X+Y)$, and deduce that V is uniformly distributed on $[0,1]$.

Solution: The transformation $x = uv$, $y = u - uv$ has Jacobian

$$J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

Therefore, we have $|J| = |u|$, and thus $f_{U,V}(u,v) = ue^{-u}$ for $0 \le u < \infty$, and $0 \le v \le 1$. Integrating with respect to u we see that we have $f_V(v) = 1$, and also that U, V are independent.

Exercise 3. A point (X,Y) is picked at random uniformly in the unit circle. Find the joint density of R and X, where $R^2 = X^2 + Y^2$.

Solution: We can make a change of variables, and use the Jacobian. We can also just compute this directly, as above, by finding the distribution function and differentiating. Using the convention that $\sqrt{r^2 - u^2} = 0$ when the argument of the square root becomes negative, we have

$$\begin{aligned} F(r,x) &= \mathbb{P}(R \le r, X \le x) = \frac{2}{\pi}\int_{-r}^{x} \sqrt{r^2 - u^2}\, du, \\ f(r,x) &= \frac{\partial^2 F}{\partial r \partial x} = \frac{2r}{\pi\sqrt{r^2 - x^2}}, \quad |x| < r < 1. \end{aligned}$$

Exercise 4. Let X_1, X_2, X_3 be independent random variables, uniformly distributed on $[0,1]$.

a. What is the probability that three rods of lengths X_1, X_2, X_3 can be used to make a triangle? (That is, that the largest one is smaller than the sum of the other two.)

b. What is the probability distribution of the second largest X_k, i.e. $X^{(2)}$.

Solution:

a. Let $M = \max\{X_1, X_2, X_3\}$. The lengths X_1, X_2, X_3 form a triangle iff $M \le X_1 + X_2 + X_3 - M$, i.e., the sum of any two sides is at least that of the third side. By symmetry, the probability that $M = X_i$ is the same for all

i, hence we have

$$\begin{aligned}\mathbb{P}(X_1, X_2, X_3 \text{ forms a triangle}) &= 3\mathbb{P}(X_1 \le X_2 + X_3, X_2 \le X_1, X_3 \le X_1) \\ &= 3\int_0^1 \int_{\{x_2+x_3\ge x_1, x_2\le x_1, x_3\le x_1\}} dx_2 dx_3 dx_1 \\ &= 3\int_0^1 \frac{x_1^2}{2} dx_1 \\ &= \frac{1}{2}.\end{aligned}$$

b. The joint PDF for the order statistics is

$$f(x^{(1)}, x^{(2)}, x^{(3)}) = n! f(x^{(1)}) f(x^{(2)}) f(x^{(3)}) \mathbb{1}_{x^{(1)}<x^{(2)}<x^{(3)}}.$$

Integrating out $x^{(1)}$ and $x^{(3)}$

$$\begin{aligned}f(x^{(2)}) &= \int_{\mathbb{R}}\int_{\mathbb{R}} 3! f(x^{(1)}) f(x^{(2)}) f(x^{(3)}) \mathbb{1}_{x^{(1)}<x^{(2)}<x^{(3)}}\, dx^{(1)}\, dx^{(3)} \\ &= 3! f(x^{(2)}) \left(\int_{x^{(2)}}^{\infty} f(x^{(1)})\, dx^{(1)}\right)\left(\int_{-\infty}^{x^{(2)}} f(x^{(3)})\, dx^{(3)}\right) \\ &= 3! f(x^{(2)})(1 - F(x^{(2)}))F(x^{(2)}).\end{aligned}$$

In particular, for X_k uniform

$$f(x^{(2)}) = 3! x^{(2)}(1 - x^{(2)}) \mathbb{1}_{[0,1]}(x^{(2)}).$$

Exercise 5. A stick is broken, at a location chosen uniformly at random. Find the average ratio of the lengths of the smaller and larger pieces.

Solution: WLOG assume the stick has unit length and by symmetry assume the small piece is distributed uniformly on $[0, \frac{1}{2}]$ with PDF

$$f_S(s) = \begin{cases} 2 & 0 \le s \le 1/2 \\ 0 & \text{else} \end{cases}.$$

Let $g : [0, 1/2] \to [0, 1]$, $g(x) = x/(1 - x)$. This function has a well defined inverse $g^{-1}(x) = x/(1 + x)$ and derivative $(g^{-1})'(x) = (1 + x)^{-2}$. Let $X =$

$g(S)$, the ratio of the small to large piece. Using the formula from lecture 12, the resulting PDF is

$$\begin{aligned} f_X(x) &= f_S(g^{-1}(x)) \frac{1}{|g'(g^{-1}(x))|} \\ &= f_S\left(\frac{x}{1+x}\right) |(g^{-1})'(x)| \\ &= 2(1+x)^{-2} \mathbb{1}_{[0,1]}(x). \end{aligned}$$

Therefore, the resulting expected ratio is

$$E[X] = \int_0^1 x 2\,(1+x)^{-2}\ dx = \log(4) - 1.$$

Exercise 6. Let $X \sim \Gamma(a, c)$, $U, V \sim \Gamma(a, \sqrt{2c})$ and $Y \sim \mathcal{N}(0, 1)$, all jointly independent. Compare the distribution of $U - V$ and $\sqrt{X}Y$. (*Hint:* compute MGFs using conditional expectation).

Solution: Let $N \sim \mathcal{N}(\mu, \sigma^2)$ and $G \sim \Gamma(a, c)$ there respective moment generating functions are

$$\begin{aligned} M_N(s) &= \exp\left(\mu s + \tfrac{\sigma^2}{2} s^2\right) \\ M_G(s) &= \int_0^\infty e^{sx} \frac{c^a x^{a-1} e^{-cx}}{\Gamma(a)}\, dx \\ &= \frac{c^a}{\Gamma(a)} \int_0^\infty x^{a-1} e^{-(c-s)x}\, dx \\ &= \quad 1 - \frac{s}{c} \quad {}^{-a} \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t}\, dt \quad (c - s > 0) \\ &= \quad 1 - \frac{s}{c} \quad {}^{-a} \quad (c - s > 0). \end{aligned}$$

Conditional on $X = x$, $\sqrt{x}Y \sim \mathcal{N}(0, x)$. Therefore, the MGF for $\sqrt{X}Y$ is

$$\begin{aligned} M_{\sqrt{X}Y}(s) &= E\left[\exp\left(s\sqrt{X}Y\right)\right] \\ &= E\left[E\left[\exp\left(s\sqrt{x}Y\right) \mid X = x\right]\right] \\ &= E\left[\exp\left(\frac{X}{2} s^2\right)\right] \\ &= E\left[\exp\left(\frac{s^2}{2} X\right)\right] \\ &= \left(1 - \frac{s^2}{2c}\right)^{-a} \quad \left(c - s^2/2 > 0\right). \end{aligned}$$

Similarly, the MGF for $U - V$ is

$$
\begin{aligned}
M_{U-V}(s) &= E\left[\exp\left(s(U-V)\right)\right] \\
&= E\left[E\left[\exp\left(s(U-v)\right) \mid V=v\right]\right] \\
&= E\left[e^{-sv} E\left[\exp\left(sU\right) \mid V=v\right]\right] \\
&= E\left[e^{-sv}\left(1-\frac{s}{\sqrt{2c}}\right)^{-a}\right] \quad \left(\sqrt{2c}-s>0\right) \\
&= \left(1-\frac{s}{\sqrt{2c}}\right)^{-a} E\left[e^{-sv}\right] \quad \left(c-s^2/2>0\right) \\
&= \left(1-\frac{s}{\sqrt{2c}}\right)^{-a}\left(1-\frac{-s}{\sqrt{2c}}\right)^{-a} \quad \left(c-s^2/2>0\right. \\
&= \left(1-\frac{s^2}{2c}\right)^{-a} \quad \left(c-s^2/2>0\right).
\end{aligned}
$$

Hence $\sqrt{X}Y$ and $U - V$ have the same MGF and thusly, the same distribution.

Exercise 7. Let $X, Y \sim \Gamma(1, c)$ be independent and $Z = X + Y$. Describe conditional distribution $P_{Y|Z}$. (Ideally, you want to describe it as a Markov kernel $K(z, dy)$, however, full credit will be given for just specifying the conditional pdf or cdf).

Solution: The PDF for a Gamma random variable $U \sim \Gamma(a, c)$ is

$$
f_U(u) = \frac{1}{\Gamma(a)} c^a u^{a-1} e^{-cu}.
$$

Moreover, for two random variables with the same scale parameter the shape parameters are additive. In particular, $Z = X + Y \sim \Gamma(2, c)$. Thus,

$$
\begin{aligned}
f_Z(z) &= \frac{1}{\Gamma(2)} c^2 z^{2-1} e^{-cz} \mathbb{1}_{[0,\infty)}(z) = c^2 z e^{-cz} \mathbb{1}_{[0,\infty)}(z) \\
f_X(t) = f_Y(t) &= \frac{1}{\Gamma(1)} c e^{-ct} = c e^{-ct} \mathbb{1}_{[0,\infty)}(t).
\end{aligned}
$$

The conditional distribution for $Y|Z$ is

$$\begin{aligned} f_{Y|Z}(y \mid z) &= \frac{f_{Z|Y}(z \mid y) f_Y(y)}{f_Z(z)} \\ &= \frac{f_X(z-y) f_Y(y)}{f_Z(z)} \\ &= \frac{ce^{-c(z-y)} ce^{-cy}}{c^2 z e^{-cz}} \mathbb{1}_{[0,\infty)}(z-y) \mathbb{1}_{[0,\infty)}(y) \mathbb{1}_{[0,\infty)}(z) \\ &= \frac{1}{z} \mathbb{1}_{[0,\infty)}(y) \mathbb{1}_{[y,\infty)}(z). \end{aligned}$$

Hence, the corresponding Markov Kernel is

$$K(z, dy) = f_{Y|Z}(y \mid z)\, dy = \frac{1}{z} \mathbb{1}_{[0,\infty)}(y) \mathbb{1}_{[y,\infty)}(z)\, dy.$$